SIMON MARAIS
MATHEMATICS COMPETITION

2017 SESSION A
INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed.
- Write your solutions in English, using a black or blue pen, on the A4/Letter paper provided. Write on **only one side** of each sheet of paper, leaving margins in case scanning is required.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "A3 P2". Each page must relate to only one problem.
- You are permitted to submit more than one attempted solution to a problem, but the pages should all be numbered in one sequence.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

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PROBLEMS

A1. The five sides and five diagonals of a regular pentagon are drawn on a piece of paper. Two people play a game, in which they take turns to colour one of these ten line segments. The first player colours line segments blue, while the second player colours line segments red. A player cannot colour a line segment that has already been coloured. A player wins if they are the first to create a triangle in their own colour, whose three vertices are also vertices of the regular pentagon. The game is declared a draw if all ten line segments have been coloured without a player winning.

Determine whether the first player, the second player, or neither player can force a win.

A2. Let $a_1, a_2, a_3, \ldots$ be the sequence of real numbers defined by $a_1 = 1$ and

$$a_m = \frac{1}{a_1^2 + a_2^2 + \cdots + a_{m-1}^2} \qquad \text{for } m \geq 2.$$

Determine whether there exists a positive integer N such that

$$a_1 + a_2 + \cdots + a_N > 2017^{2017}.$$

A3. For each positive integer n, let $M(n)$ be the $n \times n$ matrix whose (i, j) entry is equal to 1 if $i + 1$ is divisible by j, and equal to 0 otherwise.

Prove that $M(n)$ is invertible if and only if $n + 1$ is square-free.

(An integer is *square-free* if it is not divisible by the square of an integer larger than 1.)

A4. Let $A_1, A_2, \ldots, A_{2017}$ be the vertices of a regular polygon with 2017 sides.

Prove that there exists a point P in the plane of the polygon such that the vector

$$\sum_{k=1}^{2017} k \frac{\overrightarrow{PA_k}}{\|\overrightarrow{PA_k}\|^5}$$

is the zero vector.

(The notation $\|\overrightarrow{XY}\|$ represents the length of the vector $\overrightarrow{XY}$.)

