SIMON MARAIS
MATHEMATICS COMPETITION

2017 SESSION B
INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed.
- Write your solutions in English, using a black or blue pen, on the A4/Letter paper provided. Write on **only one side** of each sheet of paper, leaving margins in case scanning is required.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "B3 P2". Each page must relate to only one problem.
- You are permitted to submit more than one attempted solution to a problem, but the pages should all be numbered in one sequence.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

SIMON MARAIS MATHEMATICS COMPETITION

2017 SESSION B PROBLEMS

B1. Maryam labels each vertex of a tetrahedron with the sum of the lengths of the three edges meeting at that vertex. She then observes that the labels at the four vertices of the tetrahedron are all equal.

For each vertex of the tetrahedron, prove that the lengths of the three edges meeting at that vertex are the three side lengths of a triangle.

B2. Determine all pairs (p, q) of positive integers such that p and q are prime, and $p^{q-1} + q^{p-1}$ is the square of an integer.

B3. Each point in the plane with integer coordinates is coloured red or blue such that the following two properties hold.

- For any two red points, the line segment joining them does not contain any blue points.
- For any two blue points that are distance 2 apart, the midpoint of the line segment joining them is blue.

Prove that if three red points are the vertices of a triangle, then the interior of the triangle does not contain any blue points.

B4. *The following problem is open in the sense that no solution is currently known. Progress on the problem may be awarded points. An example of progress on the problem is a non-trivial bound on the sequence defined below.*

For each integer $n \geq 2$, consider a regular polygon with $2n$ sides, all of length 1. Let $C(n)$ denote the number of ways to tile this polygon using quadrilaterals whose sides all have length 1.

Determine the limit inferior and the limit superior of the sequence defined by

$$\frac{1}{n^2} \log_2 C(n).$$

