SIMON MARAIS

MATHEMATICS COMPETITION

2019 SESSION B: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed.
- Write your solutions in English, using a black or blue pen, on the A4/Letter paper provided. Write on **only one side** of each sheet of paper, leaving margins in case scanning is required.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "B3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

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2019 SESSION B: PROBLEMS

B1. Determine all pairs (a, b) of real numbers with $a \leq b$ that maximise the integral

$$\int_a^b e^{\cos x} \left(380 - x - x^2\right) \, \mathrm{d}x.$$

B2. For each odd prime p, prove that the integer

$$1! + 2! + 3! + \cdots + p! - \left\lfloor \frac{(p-1)!}{e} \right\rfloor$$

is divisible by p.

(Here, e denotes the base of the natural logarithm and $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.)

B3. Let G be a finite simple graph and let k be the largest number of vertices of any clique in G. Suppose that we label each vertex of G with a non-negative real number, so that the sum of all such labels is 1. Define the *value of an edge* to be the product of the labels of the two vertices at its ends. Define the *value of a labelling* to be the sum of the values of the edges.

Prove that the maximum possible value of a labelling of G is $\frac{k-1}{2k}$.

(A *finite simple graph* is a graph with finitely many vertices, in which each edge connects two distinct vertices and no two edges connect the same two vertices. A *clique* in a graph is a set of vertices in which any two are connected by an edge.)

B4. *The following problem is open in the sense that no solution is currently known to part (b). A proof of part (a) will be awarded 3 points.*

A *binary string* is a sequence, each of whose terms is 0 or 1. A set $\mathcal{B}$ of binary strings is defined inductively according to the following rules.

- The binary string 1 is in $\mathcal{B}$.
- If $s_1, s_2, \ldots, s_n$ is in $\mathcal{B}$ with n odd, then both $s_1, s_2, \ldots, s_n, 0$ and $0, s_1, s_2, \ldots, s_n$ are in $\mathcal{B}$.
- If $s_1, s_2, \ldots, s_n$ is in $\mathcal{B}$ with n even, then both $s_1, s_2, \ldots, s_n, 1$ and $1, s_1, s_2, \ldots, s_n$ are in $\mathcal{B}$.
- No other binary strings are in $\mathcal{B}$.

For each positive integer n, let b_n be the number of binary strings in $\mathcal{B}$ of length n.

(a) Prove that there exist constants $c_1, c_2 > 0$ and $1.6 < \lambda_1, \lambda_2 < 1.9$ such that $c_1 \lambda_1^n < b_n < c_2 \lambda_2^n$ for all positive integers n.

(b) Determine $\liminf_{n \to \infty} \sqrt[n]{b_n}$ and $\limsup_{n \to \infty} \sqrt[n]{b_n}$.

