SIMON MARAIS

MATHEMATICS COMPETITION

2020 SESSION B: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions.
- For participants sitting the competition remotely, an electronic device such as a PC, laptop, phone or tablet may be used during the competition to access the questions.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "B3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and session (as in **S12345_B** (for singles) or **P31416_B** (for pairs), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

SIMON MARAIS MATHEMATICS COMPETITION

2020 SESSION B: PROBLEMS

B1. Let $\mathcal{M}$ be the set of 5×5 real matrices of rank 3. Given a matrix A in $\mathcal{M}$, the set of columns of A has $2^5 - 1 = 31$ nonempty subsets. Let k_A be the number of these subsets that are linearly independent.

Determine the maximum and minimum values of k_A, as A varies over $\mathcal{M}$.

The rank *of a matrix is the dimension of the span of its columns.*

B2. For each positive integer k, let S_k be the set of real numbers that can be expressed in the form

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k},$$

where $n_1, n_2, \ldots, n_k$ are positive integers.

Prove that S_k does not contain an infinite strictly increasing sequence.

B3. A cat is trying to catch a mouse in the nonnegative quadrant

$$N = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}.$$

At time $t = 0$ the cat is at $(1, 1)$ and the mouse is at $(0, 0)$. The cat moves with speed $\sqrt{2}$ such that its position $c(t) = (c_1(t), c_2(t))$ is continuous, and differentiable except at finitely many points; while the mouse moves with speed 1 such that its position $m(t) = (m_1(t), m_2(t))$ is also continuous, and differentiable except at finitely many points. Thus

$$c(0) = (1, 1), \qquad\qquad m(0) = (0, 0);$$

$c(t)$ and $m(t)$ are continuous functions of t such that $c(t), m(t) \in N$ for all $t \geq 0$; the derivatives $c'(t) = (c_1'(t), c_2'(t))$ and $m'(t) = (m_1'(t), m_2'(t))$ each exist for all but finitely many t; and

$$(c_1'(t))^2 + (c_2'(t))^2 = 2, \qquad\qquad (m_1'(t))^2 + (m_2'(t))^2 = 1,$$

whenever the respective derivative exists.

At each time t the cat knows both the mouse's position $m(t)$ and velocity $m'(t)$. Show that, no matter how the mouse moves, the cat can catch it by time $t = 1$; that is, show that the cat can move such that $c(\tau) = m(\tau)$ for some $\tau \in [0, 1]$.

B4. *The following problem is open in the sense that no solution is currently known to part (b). A proof of part (a) will be awarded 3 points.*

Let $n \geq 2$ be an integer, and let P_n be a regular polygon with $n^2 - n + 1$ vertices. We say that n is *taut* if it is possible to choose n of the vertices of P_n such that the pairwise distances between the chosen vertices are all distinct.

(a) Show that if $n - 1$ is prime then n is taut.

(b) Which integers $n \geq 2$ are taut?