SIMON MARAIS mathematics competition

2023 PAPER A: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions. Tablets may be used solely for writing worked solutions, with internet access switched off.
- For participants sitting the exam off-site, an electronic device such as a PC, laptop, phone or tablet may be used during the competition for accessing the papers, undergoing invigilation, writing and submitting solutions and (for pairs entrants) communicating with the other member of the pair.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, “A3 P2”. Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and the paper (as in **S1234567A** (for singles) or **P3141593A** (for pairs)), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

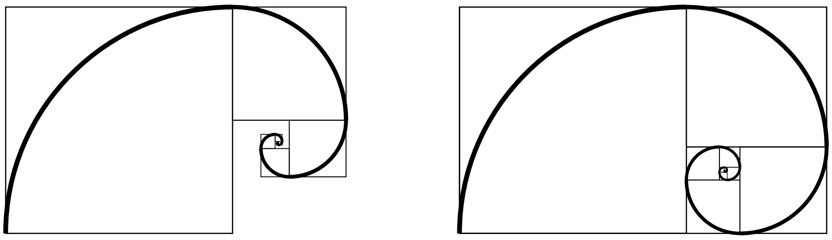
Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

SIMON MARAIS mathematics competition

2023 PAPER A: PROBLEMS

A1. For $0 < q < 1$, the *q-Fibonacci spiral* is constructed as described below. An arc of radius 1 is first drawn inside a 1×1 square. A second arc is then drawn in a $q \times q$ square, then a third in a $q^2 \times q^2$ square, and so on ad infinitum, to create a continuous curve. The example on the left shows the case $q = \frac{1}{2}$, while the example on the right shows $q = \frac{\sqrt{5}-1}{2}$.



$q = \frac{1}{2}$ $\qquad$ $q = \frac{\sqrt{5}-1}{2}$

Prove that there exists a circle centred at the centre of the initial 1×1 square such that for each $0 < q < 1$, the limiting endpoint of the q-Fibonacci spiral lies on this circle.

A2. Let n be a positive integer and let $f_1(x), \ldots, f_n(x)$ be affine functions from $\mathbb{R}$ to $\mathbb{R}$ such that, amongst the n graphs of these functions, no two are parallel and no three are concurrent. Let S be the set of all convex functions $g(x)$ from $\mathbb{R}$ to $\mathbb{R}$ such that for each $x \in \mathbb{R}$, there exists i such that $g(x) = f_i(x)$.

Determine the largest and smallest possible values of $|S|$ in terms of n.

(A function $f(x)$ is *affine* if it is of the form $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. A function $g(x)$ is *convex* if $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$ for all $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$.)

A3. For each positive integer n, let $f(n)$ denote the smallest possible value of

$$|A_1 \cup A_2 \cup \cdots \cup A_n|,$$

where $A_1, A_2, \ldots, A_n$ are sets such that $A_i \not\subseteq A_j$ and $|A_i| \neq |A_j|$ whenever $i \neq j$.

Determine $f(n)$ for each positive integer n.

A4. Let $x_0, x_1, x_2, \ldots$ be a sequence of positive real numbers such that for all $n \geq 0$,

$$x_{n+1} = \frac{(n^2+1)x_n^2}{x_n^3 + n^2}.$$

For which values of x_0 is this sequence bounded?

