SIMON MARAIS mathematics competition

2023 PAPER B: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem B1–B3 is worth 7 points. Points for problem B4 are as indicated.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions. Tablets may be used solely for writing worked solutions, with internet access switched off.
- For participants sitting the exam off-site, an electronic device such as a PC, laptop, phone or tablet may be used during the competition for accessing the papers, undergoing invigilation, writing and submitting solutions and (for pairs entrants) communicating with the other member of the pair.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "B3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and the paper (as in **S1234567B** (for singles) or **P3141593B** (for pairs)), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

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2023 PAPER B: PROBLEMS

B1. Find the smallest positive real number r with the following property: For every choice of 2023 unit vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2023} \in \mathbb{R}^2$, a point $\mathbf{p}$ can be found in the plane such that for each subset S of $\{1, 2, \ldots, 2023\}$, the sum

$$\sum_{i \in S} \mathbf{v}_i$$

lies inside the disc $\{\mathbf{x} \in \mathbb{R}^2 : ||\mathbf{x} - \mathbf{p}|| \leq r\}$.

B2. There are 256 players in a tennis tournament who are ranked from 1 to 256, with 1 corresponding to the highest rank and 256 corresponding to the lowest rank. When two players play a match in the tournament, the player whose rank is higher wins the match with probability $\frac{3}{5}$.

In each round of the tournament, the player with the highest rank plays against the player with the second highest rank, the player with the third highest rank plays against the player with the fourth highest rank, and so on. At the end of the round, the players who win proceed to the next round and the players who lose exit the tournament. After eight rounds, there is one player remaining in the tournament and they are declared the winner.

Determine the expected value of the rank of the winner.

B3. Let n be a positive integer. Let A, B and C be three n-dimensional subspaces of $\mathbb{R}^{2n}$ with the property that $A \cap B = B \cap C = C \cap A = \{\mathbf{0}\}$. Prove that there exist bases $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ of A, $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of B and $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ of C with the property that for each $i \in \{1, 2, \ldots, n\}$, the vectors $\mathbf{a}_i$, $\mathbf{b}_i$ and $\mathbf{c}_i$ are linearly dependent.

B4. *The following problem is open in the sense that the answer to part (b) is not currently known. A solution to part (a) will be awarded 7 points. Up to 7 additional points may be awarded for progress on part (b).*

(a) Let n be a positive integer that is not a perfect square. Find all pairs (a, b) of positive integers for which there exists a positive real number r, such that

$$r^a + \sqrt{n} \quad \text{and} \quad r^b + \sqrt{n}$$

are both rational numbers.

(b) Let n be a positive integer that is not a perfect square. Find all pairs (a, b) of positive integers for which there exists a real number r, such that

$$r^a + \sqrt{n} \quad \text{and} \quad r^b + \sqrt{n}$$

are both rational numbers.

