SIMON MARAIS mathematics competition

# 2025 PAPER A: INSTRUCTIONS

*Time allowed: 3 hours, with no additional reading time.*

*Each problem is worth 7 points.*

*Partial credit may be awarded for an incomplete solution or progress towards a solution.*

## Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions. Tablets may be used solely for writing worked solutions, with internet access switched off.
- For participants sitting the exam off-site, an electronic device such as a PC, laptop, phone or tablet may be used during the competition for accessing the papers, undergoing invigilation, writing and submitting solutions and (for pairs entrants) communicating with the other member of the pair.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "A3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and the paper (as in **S1234567A** (for singles) or **P3141593A** (for pairs)), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

## Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

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# 2025 PAPER A: PROBLEMS

**A1.** A function $f\colon [0,\infty) \to [0,\infty)$ is *max-poly* if there exist finitely many polynomials with real coefficients $p_1(x), p_2(x), \ldots, p_n(x)$ such that

- $p_i(x) \geq 0$ for all $x \geq 0$ and $1 \leq i \leq n$, and
- $f(x) = \max(p_1(x), p_2(x), \ldots, p_n(x))$ for all $x \geq 0$.

Determine whether the function $f(x) = \min(x, x^2)$ is max-poly.

**A2.** Let $p$ be a prime number. Given a square matrix $M$ with entries in $\mathbb{Z}/p\mathbb{Z}$, an *eigenvector* of $M$ is a non-zero vector $\mathbf{v}$ with entries in $\mathbb{Z}/p\mathbb{Z}$ such that $M\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{Z}/p\mathbb{Z}$.

For each positive integer $n$, determine the average number of eigenvectors of an $n \times n$ matrix with entries in $\mathbb{Z}/p\mathbb{Z}$.

(Here $\mathbb{Z}/p\mathbb{Z}$ refers to the integers modulo $p$.)

**A3.** Consider the matrix

$$M = \begin{bmatrix} t+1 & t \\ 1 & 1 \end{bmatrix},$$

whose entries are polynomials in the variable $t$. Let $a(t)$ denote the polynomial in the first row and first column of $M^{2025}$ and let $b(t)$ denote the polynomial in the first row and second column of $M^{2025}$.

Prove that $a(t)$ has 2025 distinct real roots $\alpha_1, \alpha_2, \ldots, \alpha_{2025}$ and that $b(t)$ has 2025 distinct real roots $\beta_1, \beta_2, \ldots, \beta_{2025}$ satisfying

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_{2025} < \beta_{2025}.$$

**A4.** Let $V$ be a finite-dimensional subspace of the real vector space of $C^\infty$ functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that for all functions $g(x) \in V$, we have $g(2x) \in V$.

Prove that every function in $V$ is a polynomial.

(A function from $\mathbb{R}$ to $\mathbb{R}$ is $C^\infty$ if its $k$-th derivative exists at every point, for all positive integers $k$.)

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