SIMON MARAIS

MATHEMATICS COMPETITION

2017

SOLUTIONS (PRELIMINARY VERSION)

This document will be updated to include alternative solutions provided by contestants, after the competition has been marked.

Problem A1

The five sides and five diagonals of a regular pentagon are drawn on a piece of paper. Two people play a game, in which they take turns to colour one of these ten line segments. The first player colours line segments blue, while the second player colours line segments red. A player cannot colour a line segment that has already been coloured. A player wins if they are the first to create a triangle in their own colour, whose three vertices are also vertices of the regular pentagon. The game is declared a draw if all ten line segments have been coloured without a player winning.

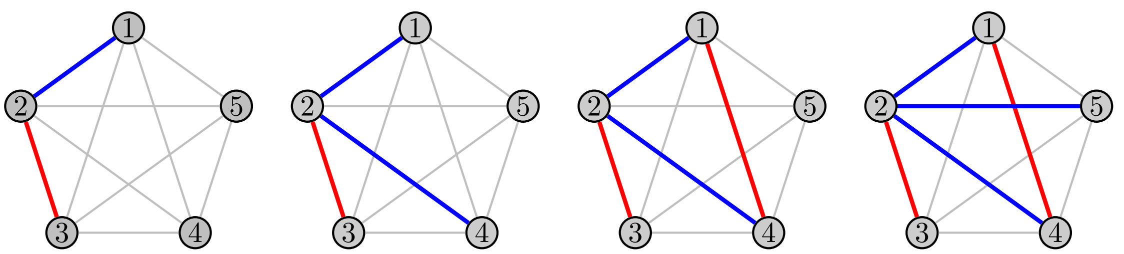
Determine whether the first player, the second player, or neither player can force a win.

Solution

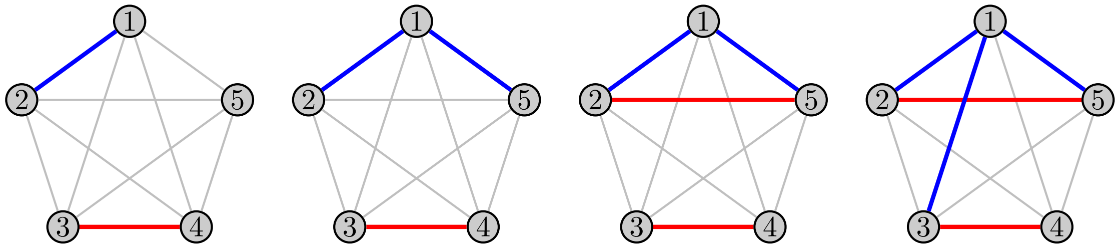
We will prove that the first player can force a win. In other words, we will prove that no matter how the second player responds, the first player can win.

Suppose that the vertices of the pentagon are P_1, P_2, P_3, P_4, P_5. Let the first player start the game by colouring the line segment P_1P_2. Without loss of generality, we may assume that the second player responds by colouring P_2P_3 or P_3P_4.

- *Case 1: The second player responds by colouring P_2P_3.*
  The first player then colours P_2P_4, forcing the second player to colour P_1P_4 to avoid losing. The first player then colours P_2P_5 and wins the game by colouring either P_1P_5 or P_4P_5 on their next move.



- *Case 2: The second player responds by colouring P_3P_4.*
  The first player then colours P_1P_5, forcing the second player to colour P_2P_5 to avoid losing. The first player then colours P_1P_3 and wins the game by colouring either P_2P_3 or P_3P_5 on their next move.



Thus, we have shown that no matter how the second player responds, the first player can win.

Comments

There are many variations and generalisations of this problem. The interested reader may like to consider the following three.

- For each integer $n \geq 3$, determine which player can force a win if the game is played on a regular n-sided polygon.

- Fix an integer $k \geq 3$. Suppose that a player wins if they are the first to colour all of the line segments $Q_1Q_2, Q_2Q_3, \ldots, Q_{k-1}Q_k, Q_kQ_1$, where $Q_1, Q_2, \ldots, Q_k$ are distinct vertices of a given polygon. For each integer $n \geq 3$, determine which player can force a win if the game is played on a regular n-sided polygon.

- Fix an integer $k \geq 3$. Suppose that a player wins if they are the first to colour all of the line segments Q_iQ_j for $1 \leq i < j \leq k$, where $Q_1, Q_2, \ldots, Q_k$ are distinct vertices of a given polygon. For each integer $n \geq 3$, determine which player can force a win if the game is played on a regular n-sided polygon.

Problem A2

Let $a_1, a_2, a_3, \ldots$ be the sequence of real numbers defined by $a_1 = 1$ and

$$a_m = \frac{1}{a_1^2 + a_2^2 + \cdots + a_{m-1}^2} \qquad \text{for } m \geq 2.$$

Determine whether there exists a positive integer N such that

$$a_1 + a_2 + \cdots + a_N > 2017^{2017}.$$

Solution

First, we note that $0 < a_m = \frac{1}{1+a_2^2+\cdots+a_{m-1}^2} \leq 1$ for all integers $m \geq 2$. One of the following two cases must arise.

- *Case 1: The sequence $a_1, a_2, a_3, \ldots$ does not converge to 0.*
  In this case, the series $a_1 + a_2 + a_3 + \cdots$ diverges to infinity.

- *Case 2: The sequence $a_1, a_2, a_3, \ldots$ converges to 0.*
  In this case, the sequence given by

  $$\frac{1}{a_m} = a_1^2 + a_2^2 + \cdots + a_{m-1}^2$$

  approaches infinity as m approaches infinity. However, the inequality $0 < a_m \leq 1$ implies that $a_m \geq a_m^2$. It follows that

  $$a_1 + a_2 + \cdots + a_{m-1} \geq a_1^2 + a_2^2 + \cdots + a_{m-1}^2 = \frac{1}{a_m}.$$

  Therefore, $a_1 + a_2 + \cdots + a_{m-1}$ approaches infinity as m approaches infinity.

In either case, we have shown that the series $a_1 + a_2 + a_3 + \cdots$ diverges to infinity. Therefore, there exists a positive integer N such that $a_1 + a_2 + \cdots + a_N > 2017^{2017}$.

Comments

The interested reader may like to prove that it is in fact Case 2 in the solution above that holds true — in other words, the sequence $a_1, a_2, a_3, \ldots$ converges to 0.

Problem A3

For each positive integer n, let $M(n)$ be the $n \times n$ matrix whose (i, j) entry is equal to 1 if $i+1$ is divisible by j, and equal to 0 otherwise.

Prove that $M(n)$ is invertible if and only if $n + 1$ is square-free.

(An integer is *square-free* if it is not divisible by the square of an integer larger than 1.)

Solution

Fix a positive integer n. For positive integers a and b, let $d(a, b) = 1$ if a is divisible by b and let $d(a, b) = 0$ otherwise. For every positive integer m, define the vector

$$v(m) = (d(m, 1), d(m, 2), \dots, d(m, n)) \in \mathbb{R}^n.$$

Observe that the rows of the matrix $M(n)$ are $v(2), v(3), \dots, v(n + 1)$. Our proof hinges on the following result.

Lemma. *If the prime factorisation of $n + 1$ is $p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, then*

$$\sum_I (-1)^{|I|}\, v\left(\frac{n+1}{\prod_{i \in I} p_i}\right) = (0, 0, \dots, 0),$$

where the summation is over all subsets of $\{1, 2, \dots, k\}$.

Proof. Let us consider the jth component of the left side of the equation, where $1 \le j \le n$.

If $n + 1$ is not divisible by j, then the jth component of each summand is 0. Hence, the jth component of the left side of the equation is 0.

If $n + 1$ is divisible by j, then we can write $\frac{n+1}{j} = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$, where $0 \le b_i \le a_i$ for each i. Let $J = \{1 \le i \le k \mid b_i > 0\}$.

Then the jth component of the left side of the equation is

$$\begin{aligned}
\sum_I (-1)^{|I|}\, d\left(\frac{n+1}{\prod_{i \in I} p_i}, \frac{n+1}{p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}}\right) &= \sum_I (-1)^{|I|}\, d\Big(p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}, \textstyle\prod_{i \in I} p_i\Big) \\
&= \sum_{I \subseteq J} (-1)^{|I|}\, d\Big(p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}, \textstyle\prod_{i \in I} p_i\Big) \\
&= \sum_{I \subseteq J} (-1)^{|I|} \\
&= 0.
\end{aligned}$$

The first equality uses the fact that $\frac{n+1}{a}$ is divisible by $\frac{n+1}{b}$ if and only if b is divisible by a. The second and third equalities use the fact that $d\Big(p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}, \prod_{i \in I} p_i\Big)$ is equal to 1 if $I \subseteq J$, and equal to 0 otherwise. The final equality uses the fact that $j \le n$, so $J \ne \emptyset$. $\square$

We now consider two cases.

- *Case 1: The integer $n+1$ is not square-free.*
  Note that the expression

  $$\sum_I (-1)^{|I|}\, v\left(\frac{n+1}{\prod_{i\in I} p_i}\right)$$

  is a linear combination of $v(1), v(2), \ldots, v(n+1)$ in which the coefficient of $v(n+1)$ is 1. Furthermore, since $n+1$ is not square-free, it must be divisible by p^2 for some prime p. It follows that the coefficient of $v(1)$ is 0.

  Therefore, one can use elementary row operations to replace the final row of the matrix $M(n)$, which is the vector $v(n+1)$, with the vector

  $$\sum_I (-1)^{|I|}\, v\left(\frac{n+1}{\prod_{i\in I} p_i}\right),$$

  without changing the determinant. By the lemma above, this is simply the zero vector. Therefore, $M(n)$ has determinant 0 and is not invertible.

- *Case 2: The integer $n+1$ is square-free.*
  Since $n+1$ is square-free, we can write $n+1 = p_1 p_2 \cdots p_k$, where $p_1, p_2, \ldots, p_k$ are distinct primes. The argument used in Case 1 does not apply in this case, since the expression

  $$\sum_I (-1)^{|I|}\, v\left(\frac{n+1}{\prod_{i\in I} p_i}\right)$$

  is a linear combination of $v(1), v(2), \ldots, v(n+1)$ with a non-zero coefficient of $v(1)$ and $v(1)$ is not a row of $M(n)$.

  Instead, we use elementary row operations to replace the final row of the matrix $M(n)$, which is $v(n+1)$, with the vector

  $$\sum_{I \neq \{1,2,\ldots,k\}} (-1)^{|I|}\, v\left(\frac{n+1}{\prod_{i\in I} p_i}\right),$$

  which does not change the determinant. By the lemma above, this is simply the vector $v(1) = (1, 0, 0, \ldots, 0)$, up to sign. Now calculate the determinant of this new matrix by cofactor expansion along this row. It tells us that the determinant of $M(n)$ is equal to its $(n, 1)$ minor, up to sign. However, the matrix obtained by deleting the nth row and 1st column of $M(n)$ is lower triangular, since $d(a, b) = 0$ for $a < b$. Moreover, it has 1s along the main diagonal, since $d(a, a) = 1$. It follows that the determinant of $M(n)$ is 1 or -1 and hence, $M(n)$ is invertible.

Comments

In the solution above, we deduced that $\det M(n) = \pm 1$ when n is square-free. The interested reader may like to determine for which positive integers n we have $\det M(n) = 1$.

Problem A4

Let $A_1, A_2, \ldots, A_{2017}$ be the vertices of a regular polygon with 2017 sides.

Prove that there exists a point P in the plane of the polygon such that the vector

$$\sum_{k=1}^{2017} k \frac{\overrightarrow{PA_k}}{\|\overrightarrow{PA_k}\|^5}$$

is the zero vector.

(The notation $\|\overrightarrow{XY}\|$ represents the length of the vector $\overrightarrow{XY}$.)

Solution

Let $P = (x, y)$ and $A = (a, b)$, and consider the following two equations.

$$\frac{\partial}{\partial x}\frac{1}{\|PA\|^3} = \frac{\partial}{\partial x}\frac{1}{[(x-a)^2+(y-b)^2]^{3/2}} = \frac{3(a-x)}{[(x-a)^2+(y-b)^2]^{5/2}} = 3\frac{a-x}{\|PA\|^5}$$
$$\frac{\partial}{\partial y}\frac{1}{\|PA\|^3} = \frac{\partial}{\partial y}\frac{1}{[(x-a)^2+(y-b)^2]^{3/2}} = \frac{3(b-y)}{[(x-a)^2+(y-b)^2]^{5/2}} = 3\frac{b-y}{\|PA\|^5}$$

Since $\overrightarrow{PA} = (a-x, b-y)$, the gradient of $\dfrac{1}{\|PA\|^3}$ as a function of P is therefore $3\dfrac{\overrightarrow{PA}}{\|\overrightarrow{PA}\|^5}$.

It follows that the function $f : \mathbb{R}^2 \setminus \{A_1, A_2, \ldots, A_{2017}\} \to \mathbb{R}$ defined by

$$f(P) = \sum_{k=1}^{2017} \frac{k}{\|\overrightarrow{PA_k}\|^3}$$

has a gradient ∇f that satisfies the equation

$$\frac{1}{3}\nabla f(P) = \sum_{k=1}^{2017} k \frac{\overrightarrow{PA_k}}{\|\overrightarrow{PA_k}\|^5}.$$

If P is a local minimum of f, then we have $\nabla f(P) = 0$. Hence, it suffices to show that there exists a local minimum of f.

Without loss of generality, suppose that the regular polygon $A_1A_2\cdots A_{2017}$ is inscribed in the unit circle centred at the origin. Let D denote the unit disk centred at the origin. Let us now consider the global minimum of f over the set $D \setminus \{A_1, A_2, \ldots, A_{2017}\}$. We claim that this global minimum is attained in the interior of D, so it must also be a local minimum of f.

It now suffices to show that $f(0,0) < f(x,y)$, for every point (x,y) on the unit circle that is not a vertex of the polygon. Indeed, we have

$$f(0,0) = \sum_{k=1}^{2017} \frac{k}{1^3} = 2\,035\,153.$$

On the other hand, if (x, y) is a point on the unit circle that is not a vertex of the polygon, then its distance to the nearest vertex of the polygon is less than the arc length $\frac{1}{2} \times \frac{2\pi}{2017} = \frac{\pi}{2017}$. Therefore, we have

$$f(x, y) = \sum_{k=1}^{2017} \frac{k}{\|\overrightarrow{PA_k}\|^3} > \frac{1}{(\pi/2017)^3} > 2\,035\,153 = f(0, 0).$$

This completes the proof.

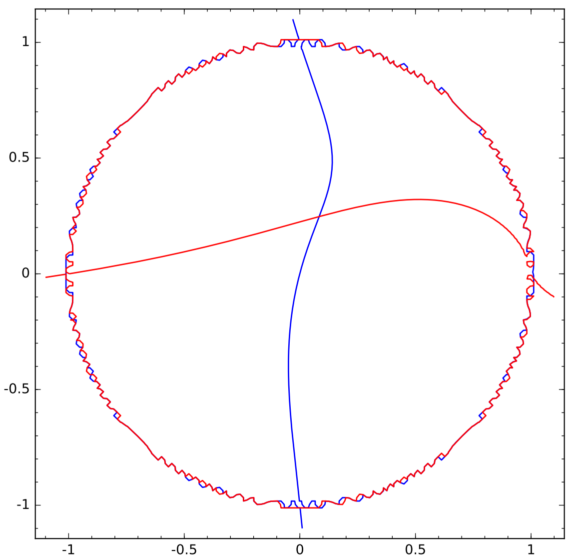
Comments

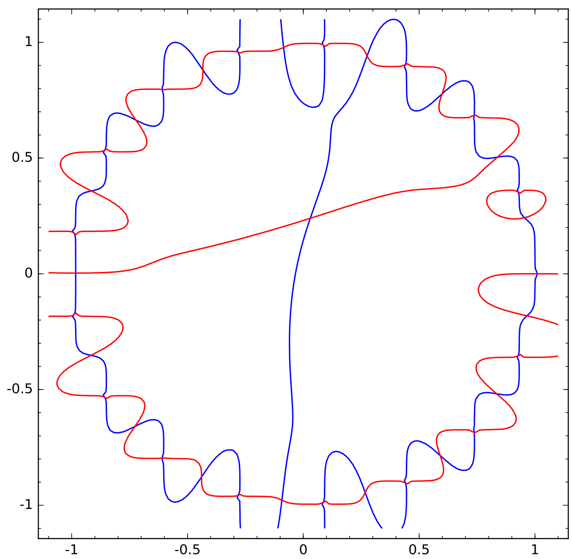
The plot below left shows in blue (respectively, red) the locus of points for which the x-coordinate (respectively, y-coordinate) of

$$\sum_{k=1}^{2017} k \frac{\overrightarrow{PA_k}}{\|\overrightarrow{PA_k}\|^5}$$

is equal to zero. The problem asks to show the existence of a point that lies in the intersection of the blue locus and the red locus. One observes that there are many such points near the unit circle and one such point closer to the origin.

The plot below right is analogous, but with all instances of the number 2017 in the original problem replaced by the number 17.





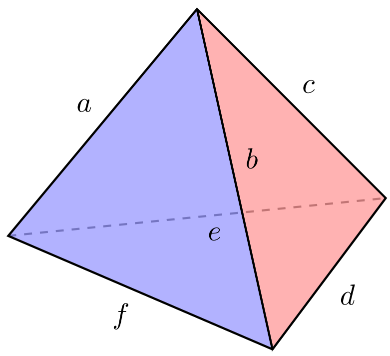
Problem B1

Maryam labels each vertex of a tetrahedron with the sum of the lengths of the three edges meeting at that vertex. She then observes that the labels at the four vertices of the tetrahedron are all equal.

For each vertex of the tetrahedron, prove that the lengths of the three edges meeting at that vertex are the three side lengths of a triangle.

Solution

Label the edge lengths of the tetrahedron a, b, c, d, e, f, where a is opposite d, b is opposite e, and c is opposite f.



These lengths must satisfy

$$a + b + c = b + d + f = c + d + e = a + f + e.$$

The first equality implies that $a + c = d + f$, while the third implies that $c + d = a + f$. Adding these two equations yields

$$a + 2c + d = a + d + 2f \qquad \Rightarrow \qquad c = f.$$

A similar argument can be used to also show that $a = d$ and $b = e$.

It follows that the lengths of the three edges meeting at any vertex are a, b and c. Furthermore, the three side lengths of any face of the tetrahedron are also a, b and c. So for each vertex of the tetrahedron, the lengths of the three edges meeting at that vertex are the three side lengths of a triangle.

Comments

The interested reader may like to consider the following problem: for which triples (a, b, c) does there exist a tetrahedron with one pair of opposite edges of length a, one pair of opposite edges of length b, and one pair of opposite edges of length c?

Problem B2

Determine all pairs (p, q) of positive integers such that p and q are prime, and $p^{q-1} + q^{p-1}$ is the square of an integer.

Solution

If $(p, q) = (2, 2)$, then $p^{q-1} + q^{p-1} = 2^1 + 2^1 = 4$, which is indeed the square of an integer. We will prove that this is the only pair of positive integers that satisfies the conditions of the problem.

If p and q are odd, then p^{q-1} and q^{p-1} are squares of odd integers, so we have

$$p^{q-1} \equiv q^{p-1} \equiv 1 \pmod 4 \qquad \Rightarrow \qquad p^{q-1} + q^{p-1} \equiv 2 \pmod 4.$$

Since the square of an integer must be congruent to 0 or 1 modulo 4, there are no pairs (p, q) that satisfy the conditions of the problem when p and q are odd.

We have deduced that a pair (p, q) that satisfies the conditions of the problem must have $p = 2$ or $q = 2$. Without loss of generality, let us assume that $q = 2$, so that we are looking for a prime p and a positive integer m such that $p + 2^{p-1} = m^2$. We have already dealt with the case $p = 2$, so let us henceforth assume that p is odd. Then we may write this equation as

$$\left(m + 2^{\frac{1}{2}(p-1)}\right)\left(m - 2^{\frac{1}{2}(p-1)}\right) = p.$$

Hence, we require $p = m + 2^{\frac{1}{2}(p-1)} > 2^{\frac{1}{2}(p-1)}$. However, we will prove below that the inequality $p > 2^{\frac{1}{2}(p-1)}$ fails to be true for $p \geq 7$. So it remains to check $p = 3$ and $p = 5$, but these two cases do not yield solutions. Therefore, we conclude that $(p, q) = (2, 2)$ is the only pair that satisfies the conditions of the problem.

For completeness, we now prove that $n \leq 2^{\frac{1}{2}(n-1)}$ for all $n \geq 7$ via induction. For $n = 7$, one can check the claim directly. Now if the claim is true for some $n \geq 7$, then it is also true for $n + 1$ since

$$n < 2^{\frac{1}{2}(n-1)} \qquad \Rightarrow \qquad n + 1 < \frac{n+1}{n} \times 2^{\frac{1}{2}(n-1)} \leq \frac{8}{7} \times 2^{\frac{1}{2}(n-1)} < \sqrt{2} \times 2^{\frac{1}{2}(n-1)} = 2^{\frac{1}{2}n}.$$

This completes the induction argument.

Comments

One can consider the same problem, but relax the condition that p and q are prime. A computer search with $1 \leq p, q \leq 100$ only reveals solutions with $p = q$. Thus, we are left with the following problem: if p and q are positive integers such that $p^{q-1} + q^{p-1}$ is the square of an integer, is it necessarily true that $p = q$?

Problem B3

Each point in the plane with integer coordinates is coloured red or blue such that the following two properties hold.

- For any two red points, the line segment joining them does not contain any blue points.
- For any two blue points that are distance 2 apart, the midpoint of the line segment joining them is blue.

Prove that if three red points are the vertices of a triangle, then the interior of the triangle does not contain any blue points.

Solution

Consider a colouring of the points in the plane with integer coordinates that satisfies the conditions of the problem. Construct the graph whose vertices are the red points, with an edge between two red points if and only if the distance between them is 1.

Lemma 1. *The graph described above is connected.*

Proof. Suppose for the sake of contradiction that the graph is not connected. Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be points from different connected components of the graph, such that the length AB is minimal. Then it is certainly the case that $AB > 1$.

There cannot exist a red point C that is in the connected component containing A as well as the connected component containing B. So by the minimality of AB, it is impossible for a red point C to simultaneously satisfy the inequalities $AC < AB$ and $BC < AB$.

The previous argument shows that the line segment AB does not contain any red points on its interior. Hence, it follows that the line segment AB does not contain any lattice points on its interior. In particular, we have deduced that A and B have different x-coordinates and different y-coordinates. Without loss of generality, let B have larger x-coordinate than A as well as larger y-coordinate than A. By the minimality of AB, the points $(a_1 + 1, a_2)$ and $(b_1, b_2 - 1)$ must be blue. So by the second condition of the problem, the points $(a_1 - 1, a_2)$ and $(b_1, b_2 + 1)$ must be red. Therefore, none of the following four midpoints are blue.

$$\left(\tfrac{a_1+b_1}{2}, \tfrac{a_2+b_2}{2}\right) \qquad \left(\tfrac{a_1+b_1+1}{2}, \tfrac{a_2+b_2}{2}\right) \qquad \left(\tfrac{a_1+b_1}{2}, \tfrac{a_2+b_2-1}{2}\right) \qquad \left(\tfrac{a_1+b_1+1}{2}, \tfrac{a_2+b_2-1}{2}\right)$$

One of these four points C is a lattice point, so it must be red. However, we now have a red point C that simultaneously satisfies the inequalities $AC < AB$ and $BC < AB$. Since this contradicts our earlier deduction, our original assumption was incorrect, and the graph is connected. $\square$

Suppose for the sake of contradiction that there exists a triangle with red vertices A, B, C, whose interior contains a blue point X. Then A, B, C are contained in some finite connected set $\mathcal{T}$ of red points, whose convex hull contains a blue point. Let $\mathcal{S}$ be a connected set of red points whose convex hull contains a blue point, such that $|\mathcal{S}|$ is minimal.

Lemma 2. *Let P be a point in $\mathcal{S}$ such that $\mathcal{S}\backslash\{P\}$ is a connected set of red points. If $A, B, C \in \mathcal{S}$ and the interior of triangle ABC contains a blue point, then P must be one of the three points A, B, C.*

Proof. We suppose the contrary in order to obtain a contradiction. Then $\mathcal{S} \setminus \{P\}$ is a connected set of red points whose convex hull contains a blue point. However, this contradicts the minimality of $|\mathcal{S}|$. □

We will use of the following two simple facts.

- Given a set of four points in the plane, there are four triangles whose vertices belong to this set. Each of these triangles is contained in the union of the other three.
- Suppose that $\mathcal{M}$ is a finite set of at least three points in the plane. The convex hull of $\mathcal{M}$ is the union of the (possibly degenerate) triangles whose vertices belong to $\mathcal{M}$.

In particular, this means that there exist $A, B, C \in \mathcal{S}$ such that the interior of triangle ABC contains a blue point.

Lemma 3. *The set $\mathcal{S}$ takes the form of a single path $R_1, R_2, \ldots, R_n$, in which R_i is connected to R_j if and only if $|i - j| = 1$.*

Proof. We suppose the contrary in order to obtain a contradiction. Then the set $\mathcal{S}$ contains three distinct points P_1, P_2, P_3 such that $\mathcal{S} \setminus \{P_i\}$ is connected for $i = 1, 2, 3$. This is impossible if these are the only points in $\mathcal{S}$, so there must be some other point $X \in \mathcal{S}$. We know that there is some triangle whose vertices are in $\mathcal{S}$ and whose interior contains a blue point. Hence, by Lemma 2, this triangle must be $P_1P_2P_3$. However, this triangle is contained in the union of the triangles XP_1P_2, XP_2P_3 and XP_3P_1. So the interior of one of these three triangles contains a blue point, which contradicts Lemma 2. □

Now let B be a blue point contained in the convex hull of $\mathcal{S}$ that maximises the area of triangle R_1BR_n. Without loss of generality, we may assume that $R_1B \geq R_nB$. We know that there is some triangle whose vertices are in $\mathcal{S}$ and whose interior contains B. So by Lemma 2, this must be $R_1R_jR_n$ for some $1 < j < n$. So B lies in the interior of one of the triangles $R_1R_2R_n$, $R_1R_2R_j$ or $R_2R_jR_n$. So again by Lemma 2, B lies in the interior of triangle $R_1R_2R_n$.

Now let X be the point such that B is the midpoint of XR_n. Since B and R_n have integer coordinates, X also has integer coordinates. Since the line segment XR_n contains a blue point, X must be blue. The triangle XR_1R_n has twice the area of triangle BR_1R_n, so the maximality of the area of triangle BR_1R_n implies that the point X lies outside the convex hull of $\mathcal{S}$. In particular, X lies outside triangle $R_1R_2R_n$, so $R_1XR_2R_n$ is a convex quadrilateral. Since $R_1B \geq R_nB$, the midpoint M of R_1R_n must satisfy $\angle R_nMB \leq 90°$. It follows that $\angle R_nR_1X \leq 90°$, so $\angle R_nR_1R_2 < 90°$ and $\angle R_2R_1X < 90°$. Since $R_1R_2 = 1$ and the four points R_1, R_2, R_n, X have integer coordinates, $R_1XR_2 \leq 45°$ and $\angle R_1R_nR_2 \leq 45°$. These equalities force $\angle XR_2R_n > 180°$, which is impossible since $R_1XR_2R_n$ is a convex quadrilateral, so we have reached the desired contradiction.

Problem B4

The following problem is open in the sense that no solution is currently known. Progress on the problem may be awarded points. An example of progress on the problem is a non-trivial bound on the sequence defined below.

For each integer $n \geq 2$, consider a regular polygon with $2n$ sides, all of length 1. Let $C(n)$ denote the number of ways to tile this polygon using quadrilaterals whose sides all have length 1.

Determine the limit inferior and the limit superior of the sequence defined by

$$\frac{1}{n^2} \log_2 C(n).$$

Comments

The first terms of the sequence $C(1), C(2), C(3), \ldots$ are

$$1, 1, 2, 8, 62, 908, 24698, 1232944, 112018190, 18410581880, 5449192389984, 2894710651370536, 2752596959306389652, 4675651520558571537540, 14163808995580022218786390, \ldots.$$

This is sequence A006245 in The On-Line Encyclopedia of Integer Sequences. It appears that no further terms are known.

The problem of studying this sequence was first proposed by Donald E. Knuth in his book *Axioms and Hulls*, published in 1992. The number $C(n)$ counts various other objects of mathematical interest, such as

- commutation classes of the longest element in the symmetric group S_n;
- heaps for the longest element in the symmetric group S_n;
- primitive sorting networks on n elements;
- uniform oriented matroids of rank three on n elements; and
- arrangements of n pseudolines.

For further information on these problems and their equivalence, we point the interested reader to the paper *On the number of commutation classes of the longest element in the symmetric group* by Hugh Denoncourt, Dana C. Ernst and Dustin Story, as well as to the references therein.

Perhaps the best bounds for $C(n)$ were obtained by Stefan Felsner and Pavel Valtr in their paper *Coding and counting arrangements of pseudolines.* They prove that

$$2^{0.1887n^2} \leq C(n) \leq 2^{0.6571n^2},$$

for sufficiently large n.