SIMON MARAIS

MATHEMATICS COMPETITION

2018 SESSION A: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed.
- Write your solutions in English, using a black or blue pen, on the A4/Letter paper provided. Write on **only one side** of each sheet of paper, leaving margins in case scanning is required.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "A3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- You are permitted to submit more than one attempted solution to a problem, but the pages should all be numbered in one sequence.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

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2018 SESSION A: PROBLEMS

A1. Call a rectangle *dominant* if it is similar to a 2×1 rectangle.

For which integers $n \geq 5$ is it possible to tile a square with n dominant rectangles, which are not necessarily congruent to each other?

A2. Ada and Byron play a game. First, Ada chooses a non-zero real number a and announces it. Then Byron chooses a non-zero real number b and announces it. Then Ada chooses a non-zero real number c and announces it. Finally, Byron chooses a quadratic polynomial whose three coefficients are a, b, c in some order.

(a) Suppose that Byron wins if the quadratic polynomial has a real root and Ada wins otherwise. Determine which player has a winning strategy.

(b) Suppose that Ada wins if the quadratic polynomial has a real root and Byron wins otherwise. Determine which player has a winning strategy.

A3. Let $y(x)$ be the unique solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \log_e \frac{y}{x}, \quad \text{where } x > 0 \text{ and } y > 0,$$

with the initial condition $y(1) = 2018$.

How many positive real numbers x satisfy the equation $y(x) = 2000$?

A4. For each positive integer n, consider a cinema with n seats in a row, numbered left to right from 1 up to n. There is a cup holder between any two adjacent seats and there is a cup holder on the right of seat n. So seat 1 is next to one cup holder, while every other seat is next to two cup holders. There are n people, each holding a drink, waiting in a line to sit down. In turn, each person chooses an available seat uniformly at random and carries out the following.

- If they sit next to two empty cup holders, then they place their drink in the left cup holder with probability $\frac{1}{2}$ or in the right cup holder with probability $\frac{1}{2}$.
- If they sit next to one empty cup holder, then they place their drink in that empty cup holder.
- If they sit next to zero empty cup holders, then they hold their drink in their hands.

Let p_n be the probability that all n people place their drink in a cup holder.

Determine $p_1 + p_2 + p_3 + \cdots$.

