SIMON MARAIS

MATHEMATICS COMPETITION

2018 SESSION B: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed.
- Write your solutions in English, using a black or blue pen, on the A4/Letter paper provided. Write on **only one side** of each sheet of paper, leaving margins in case scanning is required.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "B3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- You are permitted to submit more than one attempted solution to a problem, but the pages should all be numbered in one sequence.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

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MATHEMATICS COMPETITION

2018 SESSION B: PROBLEMS

B1. For all positive integers n and real numbers $x_1, x_2, \ldots, x_n$, prove that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \min(i,j)\, x_i x_j \geq 0.$$

(We define $\min(a,b) = a$ if $a \leq b$ and $\min(a,b) = b$ if $a > b$.)

B2. Let S be the set of real numbers that can be expressed as $\sqrt{m} - \sqrt{n}$, where m and n are positive integers.

Do there exist real numbers $a < b$ such that the open interval (a, b) contains only finitely many elements of S?

B3. Three spiders try to catch a beetle in a game. They are all initially positioned on the edges of a regular dodecahedron whose edges have length 1. At some point in time, they start moving continuously along the edges of the dodecahedron. The beetle and one of the spiders move with maximum speed 1, while the remaining two spiders move with maximum speed $\frac{1}{2018}$. Each player always knows their own position and the position of every other player. A player can turn around at any moment and can react to the behaviour of other players instantaneously. The spiders can communicate to decide on a strategy before and during the game. If any spider occupies the same position as the beetle at some time, then the spiders win the game.

Prove that the spiders can win the game, regardless of the initial positions of all players and regardless of how the beetle moves.

(A *regular dodecahedron* is a convex polyhedron with twelve faces, each of which is a pentagon with equal side lengths and equal angles. Three faces meet at each vertex.)

B4. *The following problem is open in the sense that no solution is currently known. An explicit expression with proof for the known case of $|A(5,n)|$ will be awarded 2 points. Further progress on the problem may be awarded more points.*

For positive integers m and n, let $A(m,n)$ be the set of $2 \times mn$ matrices M with entries from the set $\{1, 2, \ldots, m\}$ such that

- each of the numbers $1, 2, \ldots, m$ appears exactly $2n$ times;
- $M_{1,1} \leq M_{1,2} \leq \cdots \leq M_{1,mn}$ and $M_{2,1} \leq M_{2,2} \leq \cdots \leq M_{2,mn}$; and
- $M_{1,j} < M_{2,j}$ for $j = 1, 2, \ldots, mn$.

Determine $|A(m,n)|$.

