SIMON MARAIS

MATHEMATICS COMPETITION

2018

SOLUTIONS

This document will be updated to include alternative solutions provided by contestants, after the competition has been marked.

Problem A1

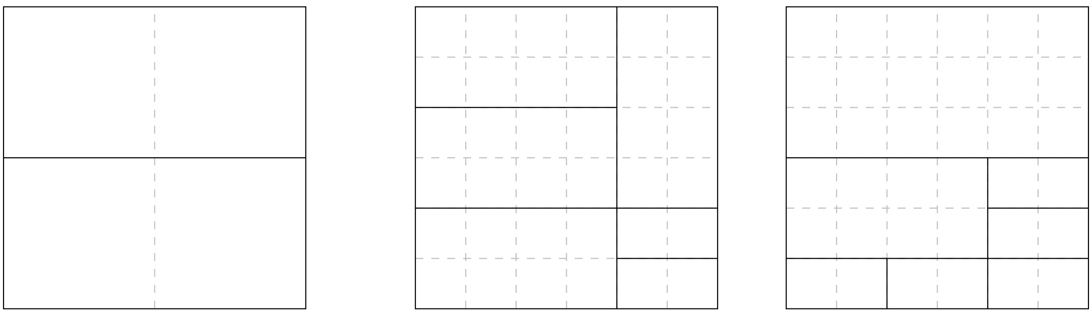
Call a rectangle *dominant* if it is similar to a 2×1 rectangle.

For which integers $n \geq 5$ is it possible to tile a square with n dominant rectangles, which are not necessarily congruent to each other?

Solution

Observe that if one can tile a square with n dominant rectangles, then one can also tile it with $n + 3$ dominant rectangles. Simply take one of the n rectangles of dimensions $2\ell \times \ell$ and divide it into four rectangles of dimensions $\ell \times \frac{1}{2}\ell$.

Now consider the following three diagrams.



- The left diagram demonstrates that one can tile the square with 2 dominant rectangles. So one can tile it with any number of dominant rectangles from the sequence $2, 5, 8, \ldots$.
- The middle diagram demonstrates that one can tile the square with 6 dominant rectangles. So one can tile it with any number of dominant rectangles from the sequence $6, 9, 12, \ldots$.
- The right diagram demonstrates that one can tile the square with 7 dominant rectangles. So one can tile it with any number of dominant rectangles from the sequence $7, 10, 13, \ldots$.

Therefore, it is possible to tile a square with n dominant rectangles for all integers $n \geq 5$.

Comments

One can show that it is impossible to tile a square with 1, 3 or 4 dominant rectangles.

A natural generalisation is to determine for which positive integers n it is possible to tile a square with n rectangles, all similar to an $a \times b$ rectangle. For fixed positive integers a and b, is the number of exceptions finite? The interested reader is invited to explore such problems.

Problem A2

Ada and Byron play a game. First, Ada chooses a non-zero real number a and announces it. Then Byron chooses a non-zero real number b and announces it. Then Ada chooses a non-zero real number c and announces it. Finally, Byron chooses a quadratic polynomial whose three coefficients are a, b, c in some order.

(a) Suppose that Byron wins if the quadratic polynomial has a real root and Ada wins otherwise. Determine which player has a winning strategy.

(b) Suppose that Ada wins if the quadratic polynomial has a real root and Byron wins otherwise. Determine which player has a winning strategy.

Solution

We may assume without loss of generality that $a = 1$.

(a) Byron has a winning strategy by choosing $b = -1$ and then choosing the quadratic polynomial $ax^2 + cx + b = x^2 + cx - 1$.

The discriminant of this quadratic is $c^2 + 4$, which is positive for any choice of c. So the quadratic polynomial has a real root.

(b) Ada has a winning strategy as follows, depending on whether Byron chooses b positive or negative.

- *Case 1.* If Byron chooses $b > 0$, then Ada can choose $c = -4\sqrt{b}$.
  Each of the six possible quadratics that can be written down has one of the following discriminants.

  $$a^2 - 4bc = 1 + 16b\sqrt{b} \qquad b^2 - 4ac = b^2 + 16\sqrt{b} \qquad c^2 - 4ab = 12b$$

  It is clear that when b is positive, all of these are positive. So no matter which quadratic Byron decides to write down, it will always have a real root. Therefore, Ada wins.

- *Case 2.* If Byron chooses $b < 0$, then Ada can choose $c = \frac{b^2}{8}$.
  Each of the six possible quadratics that can be written down has one of the following discriminants.

  $$a^2 - 4bc = 1 - \frac{b^3}{2} \qquad b^2 - 4ac = \frac{b^2}{2} \qquad c^2 - 4ab = \frac{b^4}{64} - 4b$$

  It is clear that when b is negative, all of these are positive. Therefore, for any quadratic polynomial Byron decides to write down, Ada can always ensure that it has a real root. So we have exhibited a winning strategy for Ada.

Comments

A similar game was examined in great detail in the following paper.

William Gasarch, Lawrence C. Washington and Sam Zbarsky. *The coefficient-choosing game.* arXiv:1707.04793 [math.NT]
`https://arxiv.org/abs/1707.04793`

Problem A3

Let $y(x)$ be the unique solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \log_e \frac{y}{x}, \quad \text{where } x > 0 \text{ and } y > 0,$$

with the initial condition $y(1) = 2018$.

How many positive real numbers x satisfy the equation $y(x) = 2000$?

Solution

First, note that $y'(x) > 0$ whenever $y > x$ and $y'(x) < 0$ whenever $y < x$. So $y(x)$ is increasing for $y > x$, $y(x)$ is decreasing for $y < x$, and $y'(m) = 0$ if and only if $y(m) = m$.

Second, note that

$$y''(x) = \left(\log_e \frac{y}{x}\right)' = (\log_e y - \log_e x)' = \frac{y'(x)}{y} - \frac{1}{x} = \frac{\log_e \frac{y}{x}}{y} - \frac{1}{x} \le \frac{\frac{y}{x} - 1}{y} - \frac{1}{x} = -\frac{1}{y} < 0.$$

Here, we have applied the standard inequality $\log_e(1 + r) \le r$ for $r > -1$. So $y(x)$ is a concave function.

Lemma 1. *There exists $m > 0$ such that $y(m) = m$.*

Proof. In order to obtain a contradiction, assume otherwise and observe that the initial condition $y(1) = 2018$ and the continuity of $y(x)$ imply that $y(x) > x$ for all $x > 1$. Let

$$k = \inf_{x \ge 1} \frac{y(x)}{x},$$

and note we must have $k \ge 1$. Then there exist $0 \le \epsilon < \frac{1}{2}$ and $a > 0$ such that

$$\frac{y(a)}{a} = k + \epsilon \qquad \Rightarrow \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a} = \log_e(k + \epsilon).$$

The concavity of $y(x)$ then leads to the inequality

$$y(x) \le y(a) + (x - a)\log_e(k + \epsilon) \qquad \Rightarrow \qquad \frac{y(x)}{x} \le \frac{y(a)}{x} + \frac{x - a}{x}\log_e(k + \epsilon).$$

By taking the limit as x approaches infinity, this asserts the existence of a value of x for which

$$\frac{y(x)}{x} \le \log_e(k + \epsilon) + \frac{1}{2} < k,$$

where we have again used the standard inequality $\log_e(1 + r) \le r$ for $r > -1$ and the fact that $\epsilon < \frac{1}{2}$. However, this contradicts the definition of k, so we conclude that there must exist $m > 0$ such that $y(m) = m$. $\square$

For the value of m for which $y(m) = m$, we have $y(x)$ is increasing for $x < m$ and $y(x)$ is decreasing for $x > m$, so $y(m)$ is a unique global maximum. Combined with the concavity of

$y(x)$, this implies that $y(x)$ is only defined on the open interval $(0, c)$ for some $c > 0$. Hence, there is exactly one solution to $y(x) = 2000$ on the interval (m, c).

Now let us show that there is no solution to $y(x) = 2000$ on the interval $(0, m)$. Since $y(x)$ is increasing on this interval, it suffices to show that $\lim_{x\to 0^+} y(x) > 2000$. For $0 < x < 1$, we have $y(x) < 2018$, so

$$y'(x) = \log_e \frac{y}{x} < \log_e 2018 - \log_e x.$$

It follows that

$$\lim_{x\to 0^+} y(x) = 2018 - \int_0^1 y'(x)\,\mathrm{d}x > 2018 - \int_0^1 (\log_e 2018 - \log_e x)\,\mathrm{d}x = 2017 - \log_e 2018 > 2000.$$

(This final inequality can be obtained from observing that $\log_e 2018 < \log_2 2048 = 11$.) Therefore, there is exactly one positive real number x that satisfies the equation $y(x) = 2000$.

Problem A4

For each positive integer n, consider a cinema with n seats in a row, numbered left to right from 1 up to n. There is a cup holder between any two adjacent seats and there is a cup holder on the right of seat n. So seat 1 is next to one cup holder, while every other seat is next to two cup holders. There are n people, each holding a drink, waiting in a line to sit down. In turn, each person chooses an available seat uniformly at random and carries out the following.

- If they sit next to two empty cup holders, then they place their drink in the left cup holder with probability $\frac{1}{2}$ or in the right cup holder with probability $\frac{1}{2}$.
- If they sit next to one empty cup holder, then they place their drink in that empty cup holder.
- If they sit next to zero empty cup holders, then they hold their drink in their hands.

Let p_n be the probability that all n people place their drink in a cup holder.

Determine $p_1 + p_2 + p_3 + \cdots$.

Solution

Let us set $p_0 = 1$ and observe that $p_1 = 1$. We can also calculate that $p_2 = \frac{3}{4}$. Indeed, if $n = 2$ and the first person occupies seat 1, then both people manage to place their drink in a cup holder. If the first person occupies seat 2, then with probability $\frac{1}{2}$ they will place their drink to their right, leaving the other cup holder free for the other person. So the probability of success is

$$\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}.$$

Now we derive a recurrence relation for $p_1, p_2, p_3, \ldots$. Consider the case of $n + 1$ seats and suppose that the first person to arrive occupies seat k.

- If $k = 1$, then there is only one cup holder available to the first person and thereafter, the same process starts again with the remaining n people and n seats. So in this case, the probability of success is p_n.
- If $k > 1$ and the first person places their drink in the cup holder to their left, then the $k-1$ seats to their left have access to only $k-2$ cup holders. So it is impossible for everyone to place their drink in a cup holder. So in this case, the probability of success is 0.
- If $k > 1$ and the first person places their drink in the cup holder to their right, then the $k-1$ seats to their left and the $n-k+1$ seats to their right will become two independent versions of the same processes. So in this case, the probability of success is $p_{k-1} \times p_{n-k+1}$.

Since the probability that the first person occupies a particular seat is $\frac{1}{n+1}$, we obtain the following.

$$\begin{aligned}
p_{n+1} &= \frac{1}{n+1} \times p_n + \frac{1}{n+1} \times \frac{1}{2} \times \sum_{k=2}^{n+1} p_{k-1} \times p_{n-k+1} \\
&= \frac{1}{n+1} \times p_n + \frac{1}{n+1} \times \frac{1}{2} \times \sum_{k=1}^{n} p_k \times p_{n-k} \\
&= \frac{1}{2(n+1)} p_n + \frac{1}{2(n+1)} \sum_{k=0}^{n} p_k p_{n-k}
\end{aligned}$$

Now consider the formal power series

$$P(x) = \sum_{n=0}^{\infty} p_n x^n.$$

From the recurrence relation for p_n, we know that it satisfies the following equation.

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}x} &= \sum_{n=0}^{\infty} (n+1) p_{n+1} x^n \\
&= \sum_{n=0}^{\infty} \frac{1}{2} p_n x^n + \sum_{n=0}^{\infty} \frac{1}{2} \sum_{k=0}^{n} p_k p_{n-k} x^n \\
&= \frac{1}{2} P(x) + \frac{1}{2} P(x)^2
\end{aligned}$$

This ordinary differential equation may be solved by separation of variables and combined with the initial condition $P(0) = 1$ to obtain the solution

$$P(x) = \frac{e^{x/2}}{2 - e^{x/2}}.$$

Therefore, we obtain

$$\sum_{n=1}^{\infty} p_n = P(1) - 1 = \frac{2\sqrt{e} - 2}{2 - \sqrt{e}}.$$

Comments

A formal power series that stores a sequence of numbers as its coefficients — such as $P(x)$ in the solution above — is usually referred to as a *generating function.*

The answer $\frac{2\sqrt{e}-2}{2-\sqrt{e}}$ is $3.69348450\ldots$.

Problem B1

For all positive integers n and real numbers $x_1, x_2, \ldots, x_n$, prove that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \min(i,j)\, x_i x_j \geq 0.$$

(We define $\min(a,b) = a$ if $a \leq b$ and $\min(a,b) = b$ if $a > b$.)

Solution

We will prove the identity

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \min(i,j)\, x_i x_j = \sum_{k=1}^{n} (x_k + x_{k+1} + \cdots + x_n)^2,$$

from which the statement follows immediately, since squares are non-negative.

First, consider the coefficient of x_i^2 for $1 \leq i \leq n$ on both sides of the equation. On the left side, we obtain $\min(i,i) = i$. On the right side, we observe that x_i^2 appears with coefficient 1 in summands corresponding to $1 \leq k \leq i$ and not at all in the remaining summands. Therefore, the coefficient of x_i^2 is equal to i on both sides of the equation.

Second, consider the coefficient of $x_i x_j$ for $1 \leq i < j \leq n$ on both sides of the equation. On the left side, we obtain $\min(i,j) + \min(j,i) = 2i$. On the right side, we observe that $x_i x_j$ appears with coefficient 2 in summands corresponding to $1 \leq k \leq i$ and not at all in the remaining summands. Therefore, the coefficient of $x_i x_j$ is equal to $2i$ on both sides of the equation.

Since we have considered all possible terms that can arise, the identity is proven.

Alternative solution 1

Let M_n denote the $n \times n$ matrix whose (i,j) entry is equal to $\min(i,j)$. We wish to prove that $\mathbf{x}^T M_n \mathbf{x} \geq 0$ for all vectors $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$ with real entries. Matrices that satisfy this condition are called *positive semidefinite* and it is known that a matrix is positive semidefinite if and only if all of its eigenvalues are non-negative.

We will prove that the matrix M_n is positive semidefinite for all positive integers n by induction. The base case $n = 1$ is true since the only eigenvalue of $M_1 = [\ 1\ ]$ is 1. Now suppose that M_n is positive semidefinite for some positive integer n and consider the matrix M_{n+1}. We write it as $M_{n+1} = J_{n+1} + K_{n+1}$, where J_{n+1} is the $(n+1) \times (n+1)$ matrix with all entries equal to 1, which means that K_{n+1} must take the following form.

$$K_{n+1} = \left[\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & & \\ 0 & & & & & \\ 0 & & & M_n & & \\ \vdots & & & & & \\ 0 & & & & & \end{array}\right]$$

The matrix J_{n+1} clearly has rank 1, so it has the eigenvalue 0 with multiplicity n. One can also observe that $[1, 1, 1, \ldots, 1]^T$ is an eigenvector with eigenvalue $n + 1$. Therefore, all eigenvalues of J_{n+1} are non-negative and it follows that J_{n+1} is positive semidefinite.

The matrix K_{n+1} has the eigenvalue 0, along with all of the eigenvalues of M_n. By the induction hypothesis, the eigenvalues of M_n are non-negative. Therefore, all eigenvalues of K_{n+1} are non-negative and it follows that K_{n+1} is positive semidefinite.

Now we use the fact that the sum of positive semidefinite matrices is also positive semidefinite to conclude that M_{n+1} is positive semidefinite and the induction is complete.

Alternative solution 2

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with distribution $N(0, 1)$ — in other words, normally distributed with mean 0 and variance 1. Then consider the partial sums $Y_k = X_1 + X_2 + \cdots + X_k$ for $k = 1, 2, \ldots, n$. One can check that the covariance matrix is given by

$$\operatorname{Cov}(Y_i, Y_j) = \min(i, j).$$

By the linearity of the covariance function, we have

$$\operatorname{Var}(x_1Y_1 + x_2Y_2 + \cdots + x_nY_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{Cov}(Y_i, Y_j)\, x_ix_j = \sum_{i=1}^{n}\sum_{j=1}^{n} \min(i, j)\, x_ix_j \geq 0.$$

which is non-negative since it is the variance of a random variable.

Problem B2

Let S be the set of real numbers that can be expressed as $\sqrt{m}-\sqrt{n}$, where m and n are positive integers.

Do there exist real numbers $a<b$ such that the open interval (a,b) contains only finitely many elements of S?

Solution

We will prove that every open interval (a,b) contains infinitely many elements of S. Suppose that the interval (a,b) contains a positive rational number $\frac{p}{q}$, where p and q are relatively prime positive integers. For each positive integer k, let

$$a_k = \sqrt{k^2q^2+2pk}-\sqrt{k^2q^2},$$

and observe that this number is an element of S.

We now calculate the following limit.

$$\begin{aligned}\lim_{k\to\infty} a_k &= \lim_{k\to\infty} \sqrt{k^2q^2+2pk}-\sqrt{k^2q^2} \\ &= \lim_{k\to\infty} \frac{(k^2q^2+2pk)-(k^2q^2)}{\sqrt{k^2q^2+2pk}+\sqrt{k^2q^2}} \\ &= \lim_{k\to\infty} \frac{2pk}{\sqrt{k^2q^2+2pk}+kq} \\ &= \lim_{k\to\infty} \frac{2p}{\sqrt{q^2+\frac{2p}{k}}+q} \\ &= \frac{p}{q}\end{aligned}$$

It follows that any open interval containing a positive rational number contains infinitely many elements of S. Since a number is in S if and only if its negative is in S, we also know that any open interval containing a negative rational number contains infinitely many elements of S. The statement now follows, since every open interval contains a positive rational number or a negative rational number.

Alternative solution

Let k be a positive integer such that

$$k > \frac{1}{4(b-a)^2} \qquad \Leftrightarrow \qquad \frac{1}{2\sqrt{k}} < b-a.$$

Now let m be a positive integer such that

$$m > (b+\sqrt{k})^2 \qquad \Rightarrow \qquad \sqrt{m} > b+\sqrt{k}.$$

Also, observe that the sequence $\sqrt{m}-\sqrt{1}, \sqrt{m}-\sqrt{2}, \sqrt{m}-\sqrt{3}, \ldots$ has first term greater than or equal to b and decreases to negative infinity. So there exists an integer $n \geq 2$ such that

$$\sqrt{m}-\sqrt{n-1} \geq b \qquad \text{and} \qquad \sqrt{m}-\sqrt{n} < b.$$

We claim that $\sqrt{m}-\sqrt{n} \in (a,b)$. Certainly, we have $\sqrt{m}-\sqrt{n} < b$, so it remains to prove that $\sqrt{m}-\sqrt{n} > a$. Since $\sqrt{m} > b+\sqrt{k}$, we have

$$\sqrt{m}-\sqrt{k} > b \qquad \Rightarrow \qquad \sqrt{m}-\sqrt{k} > \sqrt{m}-\sqrt{n} \qquad \Rightarrow \qquad n > k.$$

In particular, since n and k are integers, we have $n \geq k+1$ and it follows that

$$\sqrt{n}-\sqrt{n-1} = \frac{1}{\sqrt{n}+\sqrt{n-1}} \leq \frac{1}{\sqrt{k+1}+\sqrt{k}} < \frac{1}{2\sqrt{k}} < b-a.$$

Finally, we have

$$\sqrt{m}-\sqrt{n} = (\sqrt{m}-\sqrt{n-1}) - (\sqrt{n}-\sqrt{n-1}) > b-(b-a) = a,$$

which concludes the proof.

Comments

The interested reader may like to consider the following generalisation of the problem.

For each real number r, let S_r be the set of real numbers that can be expressed as $m^r - n^r$, where m and n are positive integers. Determine all values of r for which every non-empty open interval contains at least one element of S_r.

Problem B3

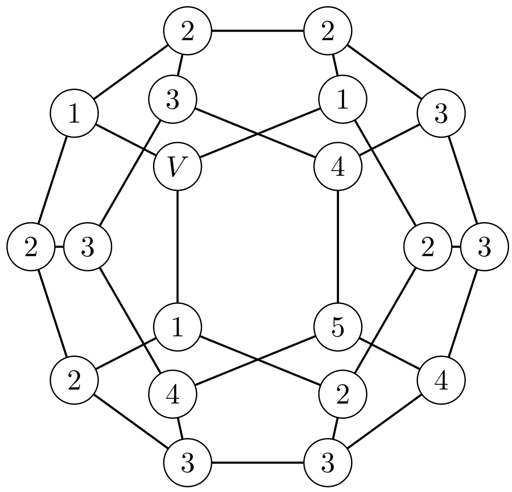
Three spiders try to catch a beetle in a game. They are all initially positioned on the edges of a regular dodecahedron whose edges have length 1. At some point in time, they start moving continuously along the edges of the dodecahedron. The beetle and one of the spiders move with maximum speed 1, while the remaining two spiders move with maximum speed $\frac{1}{2018}$. Each player always knows their own position and the position of every other player. A player can turn around at any moment and can react to the behaviour of other players instantaneously. The spiders can communicate to decide on a strategy before and during the game. If any spider occupies the same position as the beetle at some time, then the spiders win the game.

Prove that the spiders can win the game, regardless of the initial positions of all players and regardless of how the beetle moves.

(A *regular dodecahedron* is a convex polyhedron with twelve faces, each of which is a pentagon with equal side lengths and equal angles. Three faces meet at each vertex.)

Solution

Denote the graph formed by the vertices and edges of the dodecahedron by $\mathcal{G}$. Fix some vertex V of $\mathcal{G}$ and observe that there are 3 vertices at distance 1, 6 vertices at distance 2, 6 vertices at distance 3, 3 vertices at distance 4, and 1 vertex at distance 5 from V, as shown in the figure below. The distance between two vertices of $\mathcal{G}$ is defined to be the number of edges on a shortest path between them.



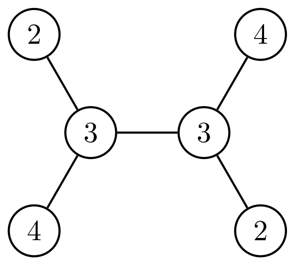
We will demonstrate a strategy that the spiders can use to win the game, regardless of the initial positions of all players and regardless of how the beetle moves.

- *Phase 1.* The fast spider moves to the vertex V and then one of the slow spiders moves to the unique vertex at distance 5 from V. Suppose this happens at time t_0.
- *Phase 2.* The other slow spider can ensure that the beetle visits a a vertex after time t_0. For if the beetle does not visit a vertex after time t_0, then it must spend all of its time on

the interior of a single edge. The other slow spider can simply walk towards that edge and then along it to force the beetle to visit a vertex after time t_0.

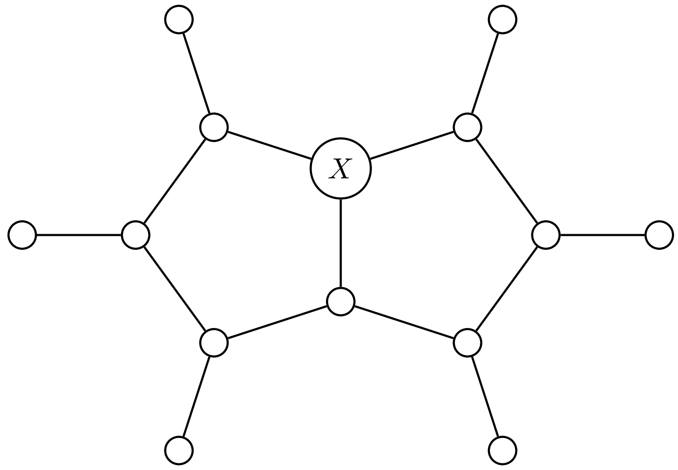
To avoid being caught by a spider, the beetle must be at a vertex at distance 1, 2, 3 or 4 from V.

- *Phase 3.* If the beetle at a vertex is at distance 1, 2 or 4 from V, then denote this vertex by W. Otherwise, the beetle is at distance 3 from V, and we will show that the slow spiders can ensure that the beetle visits a vertex at distance 2 or 4 from V. If the beetle never visits a vertex at distance 2 or 4 from V, then it is constrained to a subset of $\mathcal{G}$ that comprises two vertices at distance 3 from V, as well as the interiors of the adjacent edges, as shown in the figure below.



Since this subgraph is a tree — in other words, it is connected and has no cycles — one of the slow spiders can chase the beetle, forcing it to move to a vertex that is necessarily at distance 2 or 4 from V. Suppose this happens at time t_1 and that the vertex is W.

Let $\mathcal{P}$ be the plane that perpendicularly bisects the line segment VW. Since W is at distance 1, 2 or 4 from V, it must be the case that $\mathcal{P}$ is a plane of reflective symmetry for the dodecahedron. The set $\mathcal{G} \setminus \mathcal{P}$ consists of two connected components. Let $\mathcal{H}$ be the component that contains the vertex W, as shown in the figure below. Considering $\mathcal{H}$ as a graph, we see that it contains two cycles that share one common edge. Let X denote one of the vertices adjacent to this edge and observe that the graph $\mathcal{H} \setminus \{X\}$ is a tree.



- *Phase 4.* For all times after t_1, the fast spider moves symmetrically to the beetle, with respect to the plane $\mathcal{P}$. After time t_1, one slow spider moves to the vertex X and then stays there. Suppose this happens at time t_2.

- *Phase 5.* After time t_2, the other slow spider moves towards the beetle along a path in the tree $\mathcal{H} \setminus \{X\}$.

Now we simply observe that if the beetle meets the plane $\mathcal{P}$, then it will be captured by the fast spider there; if the beetle meets the vertex X, then it will be captured by the slow spider there; and if the beetle stays on the tree $\mathcal{H} \setminus \{X\}$, then it will be captured by the other slow spider.

Problem B4

The following problem is open in the sense that no solution is currently known. An explicit expression with proof for the known case of $|A(5,n)|$ will be awarded 2 points. Further progress on the problem may be awarded more points.

For positive integers m and n, let $A(m,n)$ be the set of $2 \times mn$ matrices M with entries from the set $\{1,2,\ldots,m\}$ such that

- each of the numbers $1,2,\ldots,m$ appears exactly $2n$ times;
- $M_{1,1} \le M_{1,2} \le \cdots \le M_{1,mn}$ and $M_{2,1} \le M_{2,2} \le \cdots \le M_{2,mn}$; and
- $M_{1,j} < M_{2,j}$ for $j = 1,2,\ldots,mn$.

Determine $|A(m,n)|$.

Solution to the case $m = 5$

Let M be a matrix satisfying the given conditions with $m = 5$ and n a positive integer. Then we must have $M_{1,i} = 1$ for $1 \le i \le 2n$ and $M_{2,j} = 5$ for $3n+1 \le j \le 5n$. Note that each matrix is uniquely determined by its first row, since the second row must consist of the remaining numbers in non-decreasing order.

Let the number of 2s, 3s and 4s in the first row be x_2, x_3 and x_4, respectively. So the number of 2s, 3s and 4s in the second row must be $2n - x_2$, $2n - x_3$ and $2n - x_4$, respectively. Therefore, we have the inequality

$$0 \le x_2, x_3, x_4 \le 2n.$$

Since 1 appears $2n$ times in the first row and 5 does not appear in the first row, we must have

$$x_2 + x_3 + x_4 = 5n - 2n = 3n.$$

Furthermore, the number of 2s and 3s in the second row must not exceed the number of 1s and 2s in the first row. So we obtain the inequality

$$2n + x_2 \ge (2n - x_2) + (2n - x_3) \qquad \Rightarrow \qquad 2x_2 + x_3 \ge 2n.$$

One can show that every integer solution to the system

$$0 \le x_2, x_3, x_4 \le 2n, \qquad x_2 + x_3 + x_4 = 3n, \qquad 2x_2 + x_3 \ge 2n,$$

leads to a a matrix satisfying the given conditions and vice versa. So $|A(5,n)|$ is precisely equal to the number of integer solutions to the system.

If we fix $x_2 = k$ for $0 \le k \le 2n$, then the system becomes

$$0 \le x_3, x_4 \le 2n, \qquad x_3 + x_4 = 3n - k, \qquad x_3 \ge 2n - 2k.$$

Using $0 \le x_4 \le 2n$ in the equation $x_3 + x_4 = 3n - k$ leads to $n - k \le x_3 \le 3n - k$. Therefore, we obtain

$$\max(0, n-k, 2n-2k) \le x_3 \le \min(2n, 3n-k),$$

and each such x_3 leads to a unique value for x_4.

We now complete the calculation of $|A(5,n)|$ as follows.

$$\begin{aligned}
|A(5,n)| &= \sum_{k=0}^{2n} [\min(2n, 3n-k) - \max(0, n-k, 2n-2k) + 1] \\
&= \sum_{k=0}^{n} [(2n) - (2n-2k) + 1] + \sum_{k=n+1}^{2n} [(3n-k) - (0) + 1] \\
&= \sum_{k=0}^{n} (2k+1) + \sum_{k=n+1}^{2n} (3n-k+1) \\
&= \frac{(n+1)(2n+2)}{2} + \frac{n(3n+1)}{2} \\
&= \frac{1}{2}(5n^2 + 5n + 2)
\end{aligned}$$

Comments

This combinatorics problem arises naturally in the study of certain spaces that arise in algebraic geometry — namely, quotients of Grassmannians by torus actions. For some related combinatorics problems, please see sequence A060854 at the *The On-Line Encyclopedia of Integer Sequences* (`https://oeis.org/A060854`).