SIMON MARAIS

MATHEMATICS COMPETITION

2019 SESSION A: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed.
- Write your solutions in English, using a black or blue pen, on the A4/Letter paper provided. Write on **only one side** of each sheet of paper, leaving margins in case scanning is required.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "A3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

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2019 SESSION A: PROBLEMS

A1. Consider the sequence $s_1, s_2, s_3, \ldots$ of positive integers defined by

- $s_1 = 2$, and
- for each positive integer n, s_{n+1} is equal to s_n plus the product of the prime factors of s_n.

The first terms of the sequence are $2, 4, 6, 12, 18, 24$.

Prove that the product of the 2019 smallest primes is a term of the sequence.

A2. Consider the operation $*$ that takes a pair of integers and returns an integer according to the rule

$$a * b = a \times (b + 1).$$

(a) For each positive integer n, determine all permutations $a_1, a_2, \ldots, a_n$ of the set $\{1, 2, \ldots, n\}$ that maximise the value of

$$(\cdots((a_1 * a_2) * a_3) * \cdots * a_{n-1}) * a_n.$$

(b) For each positive integer n, determine all permutations $b_1, b_2, \ldots, b_n$ of the set $\{1, 2, \ldots, n\}$ that maximise the value of

$$b_1 * (b_2 * (b_3 * \cdots * (b_{n-1} * b_n) \cdots)).$$

A3. For some positive integer n, a coin will be flipped n times to obtain a sequence of n heads and tails. For each flip of the coin, there is probability p of obtaining a head and probability $1 - p$ of obtaining a tail, where $0 < p < 1$ is a rational number.

Kim writes all 2^n possible sequences of n heads and tails in two columns, with some sequences in the left column and the remaining sequences in the right column. Kim would like the sequence produced by the coin flips to appear in the left column with probability $\frac{1}{2}$.

Determine all pairs (n, p) for which this is possible.

A4. Suppose that $x_1, x_2, x_3, \ldots$ is a strictly decreasing sequence of positive real numbers such that the series $x_1 + x_2 + x_3 + \cdots$ diverges.

Is it necessarily true that the series

$$\sum_{n=2}^{\infty} \min\left\{x_n, \frac{1}{n \log n}\right\}$$

diverges?

