SIMON MARAIS MATHEMATICS COMPETITION

2020 SESSION A: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions.
- For participants sitting the competition remotely, an electronic device such as a PC, laptop, phone or tablet may be used during the competition to access the questions.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "A3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and session (as in **S12345_A** (for singles) or **P31416_A** (for pairs), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

SIMON MARAIS

MATHEMATICS COMPETITION

2020 SESSION A: PROBLEMS

A1. There are 1001 points in the plane such that no three are collinear. The points are joined by 1001 line segments such that each point is an endpoint of exactly two of the line segments.

Prove that there does not exist a straight line in the plane that intersects each of the 1001 line segments in an interior point.

An interior point *of a line segment is a point of the line segment that is not one of the two endpoints.*

A2. Fiona has a deck of cards labelled 1 to n, laid out in a row on the table in order from 1 to n from left to right. Her goal is to arrange them into a single pile, through a series of steps of the following form:

> If at some stage the cards are in m piles, she chooses $1 \le k < m$ and arranges the cards into k piles by picking up pile $k+1$ and putting it on pile 1; picking up pile $k+2$ and putting it on pile 2; and so on, working from left to right and cycling back through as necessary.

She repeats this process until the cards are all in a single pile, and then stops. So for example, if $n = 7$ and she chooses $k = 3$ at the first step she will have the following three piles:

7		
4	5	6
1	2	3

If she then chooses $k = 1$ at the second step, she finishes with the cards in a single pile with the cards ordered 6352741 from top to bottom.

How many different final piles can Fiona end up with?

A3. Determine the set of all real numbers α that can be expressed in the form

$$\alpha = \sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3},$$

where $x_0, x_1, x_2, \ldots$ is an increasing sequence of real numbers with $x_0 = 1$.

A4. A *regular spatial pentagon* consists of five points P_1, P_2, P_3, P_4, P_5 in $\mathbb{R}^3$ such that $|P_iP_{i+1}| = |P_jP_{j+1}|$ and $\angle P_{i-1}P_iP_{i+1} = \angle P_{j-1}P_jP_{j+1}$ for all $1 \le i, j \le 5$, where $P_0 = P_5$ and $P_6 = P_1$. A regular spatial pentagon is *planar* if there is a plane passing through all five points P_1, P_2, P_3, P_4, P_5.

Show that every regular spatial pentagon is planar.

