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MATHEMATICS COMPETITION

2020

SOLUTIONS

Problem A1

There are 1001 points in the plane such that no three are collinear. The points are joined by 1001 line segments such that each point is an endpoint of exactly two of the line segments.

Prove that there does not exist a straight line in the plane that intersects each of the 1001 line segments in an interior point.

An interior point *of a line segment is a point of the line segment that is not one of the two endpoints.*

Solution

We present several solutions, which all hinge on the fact that 1001 is odd. Common to all solutions is an argument addressing the case of a line passing through one of the given points, so we handle this case first.

Let P be one of the given points, and suppose that L is a line passing through P. Let PA and PB be the two line segments with P as an endpoint. Then L intersects PA in an interior point if and only if A also lies on L; and similarly, L intersects PB in an interior point if and only if B also lies on L. Since P, A and B are not collinear at most one of A and B can lie on L, so L intersects at most one of PA and PB in an interior point.

Solution 1.

Suppose for the sake of contradiction that there does exist a line L intersecting each of the 1001 line segments in an interior point. We know from above that L does not pass through any of the given points, so we may colour the points on one side of L red, and the points on the other side of L blue. In order for L to intersect each of the 1001 line segments in an interior point, each of the line segments must have one red endpoint and one blue. Since each of the given points is an endpoint of exactly two of the line segments, the total number of line segments is equal to twice the number of red points. But this means that the total number of line segments must be even, contradicting the fact that

1001 is odd. We conclude that the line L does not exist.

Solution 2.

Let G be the graph whose vertices are the 1001 given points, and whose edges are the 1001 line segments. From the given conditions each vertex has degree 2, so each connected component of G is a cycle. Since the number of vertices of G is odd, there must exist a component C of G with an odd number of vertices. Label the vertices of C consecutively as $P_1, P_2, \ldots, P_{2k+1}$, so that P_i is joined to P_{i-1} and P_{i+1} for $1 \leq i \leq 2k+1$, where $P_0 = P_{2k+1}$ and $P_{2k+2} = P_1$.

Suppose now that L is a line meeting each of the 1001 line segments in an interior point. As shown above L does not pass through any of the P_i, so in order for L to intersect P_iP_{i+1} in an interior point, the points P_i and P_{i+1} must lie on opposite sides of L. It follows that $P_1, P_3, \ldots, P_{2k+1}$ must all lie on one side of L, and $P_2, P_4, \ldots, P_{2k}$ on the other. But then L does not intersect $P_{2k+1}P_1$ at all, which is a contradiction. We conclude that the line L does not exist.

Solution 3.

Consider again the graph G of Solution 2, and again let C be a connected component of G with an odd number of vertices. We may regard C as a (possibly self-intersecting) closed curve in the plane; then, for each point Q not on C we have a well defined winding number of C around Q. The winding number is constant on connected components of $\mathbb{R}^2 - C$, and we colour each connected component red if the winding number around a point in the component is even, and blue if the winding number is odd. This colouring is a proper 2-colouring of $\mathbb{R}^2 - C$, in the sense that if two connected components border along a line segment then they will be different colours. This is because the winding number changes by ± 1 whenever C is crossed away from a self-intersection, and so changes parity.

Now let L be a line in the plane that does not pass through any of the vertices of C, and consider a point Q moving along L, from one end at infinity to the other. Clearly, Q starts and finishes in a red region of the plane, because the winding number around a point outside the convex hull of C is necessarily 0. Since the colouring of $\mathbb{R}^2 - C$ is a proper colouring, Q changes colour each time it crosses an edge of C. Here we count edge crossings with multiplicities: if Q passes through a self intersection of C where k edges intersect, then we consider C to have crossed k edges, and so to have changed colour k times. Figure 1(a) and Figure 1(b) illustrate the cases $k = 3$ and $k = 4$.

Since Q starts and finishes in a red region it must change colour an even number of times, and so crosses an even number of edges belonging to C. But C has an odd number of edges, so L must fail to intersect at least one edge of C.

Comment. Since C can have only finitely many self-intersections, there is an arbitrarily small perturbation transforming L into a line L' that intersects the same line segments as L but does not pass through any self-intersection of C. By working with L' instead of

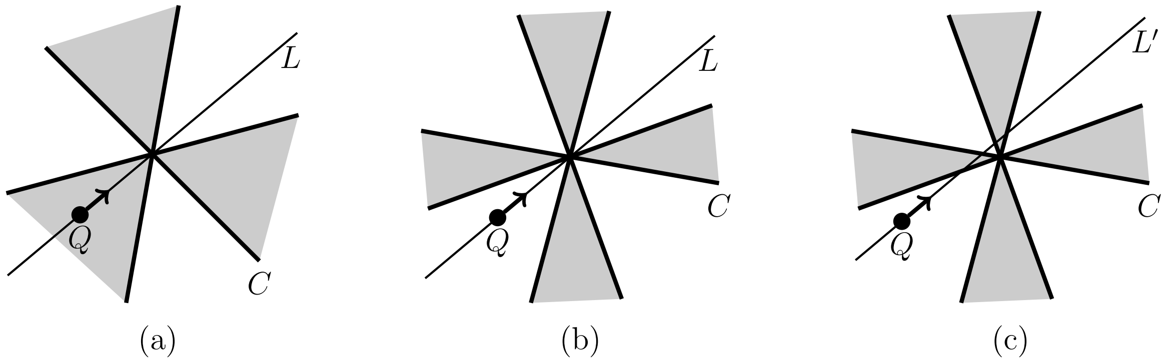

Figure 1: Effect on the colouring when Q passes through a self intersection of C. See Solution 3. (a) When Q passes through an intersection of an odd number of line segments belonging to C, the colour of the region it is in changes. (b) When Q passes through an intersection of an even number of line segments belonging to C, the colour of the region it is in remains the same. (c) Alternatively, after an arbitrarily small perturbation of L we may assume that Q does not pass through any self intersection of C, and so only ever crosses one line segment at a time.

L we could therefore assume that Q only ever passes through one edge of C at a time. This is illustrated in Figure 1(c), which is a perturbation of Figure 1(b).

Solution 4.

Let L be a line that does not pass through any of the given points. We will show that L intersects an even number of the given line segments — and so fails to intersect at least one of them — by rotating L around a suitably chosen point R on L.

Let R be a point on L such that R lies outside the convex hull of the given points, and does not lie on any line passing through two of the given points. Since R lies outside the convex hull, there is a line L_0 passing through R that does not meet any of the given line segments. In particular, L_0 intersects an even number of the given line segments.

The line L may be obtained from L_0 by rotating it about R. As the line rotates, the number of line segments it intersects remains constant except when it rotates past one of the 1001 given points. By our choice of R, the line only ever rotates past one of the given points at a time. We consider how the number of intersections with the line segments can change when the line rotates past one of the given points P. There are three cases:

(a) *Prior to rotating past P the line intersects neither of the line segments with end point P.* After rotating past P the line will intersect both line segments with endpoint P, as shown in Figure 2(a). The total number of intersections therefore changes by $+2$.

(b) *Prior to rotating past P the line intersects exactly one of the line segments with end point P.* After rotating past P the line will intersect the other line segment with

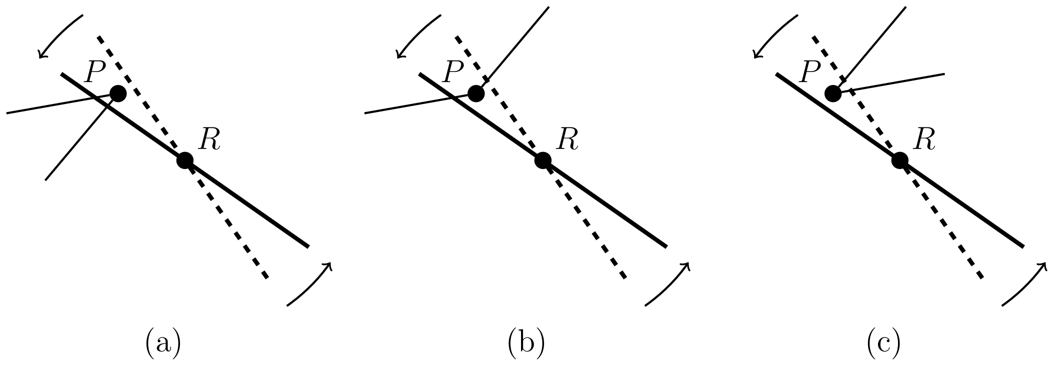

Figure 2: Effect on the number of intersections with the line segments as the line rotates around R. See Solution 4. In each figure the dashed line is the line just prior to rotating anticlockwise past P, and the solid line is the line just after.

endpoint P instead, as shown in Figure 2(b). The total number of intersections therefore does not change.

(c) *Prior to rotating past P the line intersects both of the line segments with end point P.* After rotating past P the line will intersect neither line segment with endpoint P, as shown in Figure 2(c). The total number of intersections therefore changes by -2.

In all three cases the number of intersections changes by an even number, so the parity of the number of intersections remains the same throughout the rotation. Since L_0 intersections an even number of the line segments, the line L does too.

Problem A2

Fiona has a deck of cards labelled 1 to n, laid out in a row on the table in order from 1 to n from left to right. Her goal is to arrange them into a single pile, through a series of steps of the following form:

> If at some stage the cards are in m piles, she chooses $1 \leq k < m$ and arranges the cards into k piles by picking up pile $k+1$ and putting it on pile 1; picking up pile $k+2$ and putting it on pile 2; and so on, working from left to right and cycling back through as necessary.

She repeats this process until the cards are all in a single pile, and then stops. So for example, if $n = 7$ and she chooses $k = 3$ at the first step she will have the following three piles:

7		
4	5	6
1	2	3

If she then chooses $k = 1$ at the second step, she finishes with the cards in a single pile with the cards ordered 6352741 from top to bottom.

How many different final piles can Fiona end up with?

Solution

The answer is 1 if $n = 1$, and 2^{n-2} for $n \geq 2$.

We present two solutions. The key step in both is the following lemma:

Lemma. *Let* $\mathbf{k} = (k_1, k_2, \ldots, k_\ell)$ *be a sequence of integers that Fiona may choose that results in the cards being arranged into a single pile, and let* $\pi = \pi(\mathbf{k}) = \pi_n \pi_{n-1} \cdots \pi_2 \pi_1$ *be the resulting permutation of* $\{1, 2, \ldots, n\}$*, where* π_n *is the card at the top of the pile and* π_1 *is the card at the bottom. Then*

$$k_1 = \pi_2 - \pi_1 = \pi_2 - 1.$$

Hence k_1 *is completely determined by* $\pi(\mathbf{k})$*.*

Proof. After the first move Fiona has k_1 piles, and the bottom two cards of the first pile are 1 and $k_1 + 1$, with card $k_1 + 1$ on top of card 1. The cards in the first pile don't move at all throughout the rest of the process, so at the very end we have $\pi_1 = 1$ and $\pi_2 = k_1 + 1$. It follows that

$$\pi_2 - \pi_1 = \pi_2 - 1 = (k_1 + 1) - 1 = k_1,$$

as claimed. □

Solution 1.

Let $p(n)$ be the desired sequence. Then $p(1) = 1$ (the cards are in a single pile from the start) and $p(2) = 1$ (the only possible sequence of moves is $\mathbf{k} = (1)$, which results in $\pi = 21$). Notice that $1 = 2^0 = 2^{2-2}$. We prove by induction that $p(n) = 2^{n-2}$ for $n \geq 3$ also.

Suppose that $p(m) = 2^{m-2}$ for $2 \leq m < n$, and consider the game played with n cards. After Fiona's first move k_1 she has a total of k_1 piles, and from now on the game proceeds as if each pile is a single card, numbered 1 to k_1 in order from left to right. The piles may therefore be arranged into a total of $p(k_1)$ different final piles. By the lemma k_1 is completely determined by the final outcome, so the piles we get from different choices of k_1 are all different. Hence

$$\begin{aligned} p(n) &= \sum_{k=1}^{n-1} p(k) \\ &= p(1) + \sum_{k=2}^{n-1} p(k) \\ &= 1 + \sum_{k=2}^{n-1} 2^{k-2} && \text{(by the inductive hypothesis)} \\ &= 1 + (2^{n-2} - 1) = 2^{n-2}, \end{aligned}$$

as claimed.

Solution 2.

As above there is only 1 possible outcome when $n = 1$. So we assume in what follows that $n \geq 2$.

Let $\mathbf{k} = (k_1, k_2, \ldots, k_\ell)$ be a sequence of integers that Fiona may choose that results in the cards being arranged into a single pile. Then $n > k_1 > k_2 > \cdots > k_\ell = 1$, so $K = \{k_1, \ldots, k_\ell\}$ is a subset of $\{1, 2, \ldots, n-1\}$ that contains 1. Conversely, if K is a subset of $\{1, 2, \ldots, n-1\}$ containing 1 then there is a unique ordering of the elements of K that gives a sequence of legal moves resulting in a single pile, namely ordering them from largest to smallest. Since there are 2^{n-2} such subsets, it suffices to show that $\mathbf{k}$ is completely determined by $\pi(\mathbf{k})$.

By the lemma k_1 may be determined from $\pi(\mathbf{k})$. Suppose that we have determined $k_1, k_2, \ldots, k_j$ from $\pi(\mathbf{k})$ for some $j \geq 1$. After carrying out these j moves Fiona will have a total of k_j piles, and since we know $k_1, k_2, \ldots, k_j$ we can reconstruct the order of the cards in each pile. From now on the game proceeds as if each pile is a single card; applying the lemma to the game played with k_j cards and move sequence $\mathbf{k}' = (k_{j+1}, \ldots, k_\ell)$ we may therefore recover k_{j+1} from the final pile. This completes the induction.

Problem A3

Determine the set of all real numbers α that can be expressed in the form

$$\alpha = \sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3},$$

where $x_0, x_1, x_2, \ldots$ is an increasing sequence of real numbers with $x_0 = 1$.

Solution

Let A be the set of all such real numbers, and let I be the interval

$$I = \left\{ \alpha \in \mathbb{R} : \alpha \geq \tfrac{3\sqrt{3}}{2} \right\}.$$

We show that $A = I$.

We first show that $I \subseteq A$. Let $r > 1$ be a real number, and consider the sequence $x_n = r^n$ for all $n \geq 0$. For this sequence we have

$$\sum_{n=0}^{\infty} \frac{r^{n+1}}{(r^n)^3} = \sum_{n=0}^{\infty} r^{1-2n} = \frac{r}{1-r^{-2}} = \frac{r^3}{r^2-1}. \tag{1}$$

Let $f : (1, \infty) \to \mathbb{R}$ be the function defined by $f(x) = \frac{x^3}{x^2-1}$. Then

$$f'(x) = \frac{x^2(x^2-3)}{(x^2-1)^2},$$

from which it follows that f is decreasing on $(1, \sqrt{3}]$ and increasing on $[\sqrt{3}, \infty)$. Since $f(\sqrt{3}) = \frac{3\sqrt{3}}{2}$ and

$$\lim_{r \to 1^+} f(r) = \lim_{r \to \infty} f(r) = \infty,$$

it follows that the image of f is equal to I. Equation (1) shows that $f(r) \in A$ for all $r > 1$, so $I \subseteq A$.

To prove the reverse inclusion, it suffices to show that $M = \inf A$ satisfies $M \geq \frac{3\sqrt{3}}{2}$. We present several ways to do this.

Method 1a.

Given an increasing sequence $x_0, x_1, x_2, \ldots$ with $x_0 = 1$, let

$$y_n = \frac{x_{n+1}}{x_1}$$

for all $n \geq 0$. Then $y_0, y_1, y_2, \ldots$ is an increasing sequence with $y_0 = 1$, and

$$\sum_{n=0}^{\infty} \frac{y_{n+1}}{y_n^3} = \sum_{n=1}^{\infty} \frac{y_n}{y_{n-1}^3} = \sum_{n=1}^{\infty} \frac{x_{n+1}/x_1}{(x_n/x_1)^3}$$

$$= x_1^2 \sum_{n=1}^{\infty} \frac{x_{n+1}}{x_n^3}$$

$$= x_1^2 \left(\sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3} - x_1 \right). \tag{2}$$

Given $\varepsilon > 0$ there exists a sequence $x_0, x_1, x_2, \ldots$ such that $\sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3} \leq M + \varepsilon$. Equation (2) gives

$$M \leq \sum_{n=0}^{\infty} \frac{y_{n+1}}{y_n^3} \leq x_1^2 \left(\sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3} - x_1 \right) \leq x_1^2((M+\varepsilon) - x_1),$$

so $M(x_1^2 - 1) \geq x_1^3 - \varepsilon x_1^2$. We may assume that $x_1 > 1$, because otherwise we may replace $x_0, x_1, x_2, \ldots$ with the sequence $z_0, z_1, z_2, \ldots$ defined by $z_n = x_{n+1}$, which satisfies

$$\sum_{n=0}^{\infty} \frac{z_{n+1}}{z_n^3} = \sum_{n=0}^{\infty} \frac{x_{n+2}}{x_{n+1}^3} = \sum_{n=1}^{\infty} \frac{x_{n+1}}{x_n^3} = \left(\sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3} \right) - 1.$$

Then

$$\begin{aligned} M &\geq \frac{x_1^3 - \varepsilon x_1^2}{x_1^2 - 1} \\ &\geq \frac{x_1^3 - \varepsilon x_1^3 + \varepsilon x_1^3 - \varepsilon x_1^2}{x_1^2 - 1} \\ &= (1-\varepsilon)\frac{x_1^3}{x_1^2 - 1} + \frac{\varepsilon x_1^2}{x_1 + 1} \\ &\geq (1-\varepsilon) f(x_1) \\ &\geq (1-\varepsilon)\frac{3\sqrt{3}}{2}. \end{aligned}$$

Thus $M \geq (1-\varepsilon)\frac{3\sqrt{3}}{2}$ holds for any $\varepsilon > 0$, so $M \geq \frac{3\sqrt{3}}{2}$, as claimed.

Method 1b.

Given an increasing sequence $x_0, x_1, x_2, \ldots$ with $x_0 = 1$, define the sequence $y_0, y_1, y_2, \ldots$ as above. Equation (2) rearranges to

$$\sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3} = \frac{1}{x_1^2} \left(\sum_{n=0}^{\infty} \frac{y_{n+1}}{y_n^3} + x_1^3 \right) \geq \frac{M + x_1^3}{x_1^2}.$$

By the arithmetic mean-geometric mean inequality

$$M + x_1^3 = M + \frac{x_1^3}{2} + \frac{x_1^3}{2} \geq 3 \left(\frac{M x_1^6}{4} \right)^{\frac{1}{3}} = 3x_1^2 \left(\frac{M}{4} \right)^{\frac{1}{3}},$$

so

$$\sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3} \geq 3 \left(\frac{M}{4} \right)^{\frac{1}{3}}.$$

Therefore $3\left(\frac{M}{4}\right)^{\frac{1}{3}}$ is a lower bound for A, so

$$M \geq 3\left(\frac{M}{4}\right)^{\frac{1}{3}}.$$

Rearranging we get $M \geq \frac{3\sqrt{3}}{2}$, as required.

Method 2.

Consider the function

$$g(x) = \frac{3\sqrt{3}}{2x^2}.$$

We claim that for $x > 0$ the function g is a solution to the functional equation

$$g(x) = \inf_{y>x}\left(\frac{y}{x^3} + g(y)\right).$$

Indeed, if we let $h_x(y) = \frac{y}{x^3} + g(y) = \frac{y}{x^3} + \frac{3\sqrt{3}}{2y^2}$ with x fixed we have

$$h_x'(y) = \frac{1}{x^3} - \frac{3\sqrt{3}}{y^3} = \frac{y^3 - 3\sqrt{3}x^3}{x^3y^3},$$

so h_x is decreasing on $(0, \sqrt{3}x]$, increasing on $[\sqrt{3}x, \infty)$, and has a unique critical point at $y = \sqrt{3}x > x$, at which point $h_x(\sqrt{3}x) = \frac{3\sqrt{3}}{2x^2} = g(x)$.

Now let $x_0, x_1, x_2, \ldots$ be an increasing sequence with $x_0 = 1$. If the sequence (x_n) is bounded then the series $\sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3}$ diverges, so we may assume that the sequence is unbounded. Using the definition of g, we have

$$\begin{aligned}
\frac{3\sqrt{3}}{2} &= g(x_0) \\
&\leq \frac{x_1}{x_0^3} + g(x_1) \\
&\leq \frac{x_1}{x_0^3} + \frac{x_2}{x_1^3} + g(x_2) \\
&\vdots \\
&\leq \sum_{n=0}^{m} \frac{x_{n+1}}{x_n^3} + g(x_{m+1})
\end{aligned}$$

for all m. Since (x_n) is unbounded, as m goes to infinity, $g(x_m)$ goes to 0. Thus

$$\frac{3\sqrt{3}}{2} \leq \sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3},$$

as required.

Comment: The choice of $g(x) = 3\sqrt{3}/(2x^2)$ can be motivated from the geometric series. One can also motivate it by guessing there is some lower bound to the problem, and hence trying the recursive equation

$$g(x) = \inf_{y>x}\left(\frac{y}{x^3} + g(y)\right).$$

Differentiating yields

$$g'(x) = -\frac{3y(x)}{x^4} \qquad \text{and} \qquad \frac{1}{x^3} + g'(y(x)) = 0,$$

where $y(x)$ is the argmin of the infimum given x. These combine to give

$$y(x)^4 = 3y(y(x))x^3.$$

Guessing that this has a linear solution of the form $y(x) = cx$ we may substitute to find $c = \sqrt{3}$, which in turn gives $g(y) = 3\sqrt{3}/(2x^2)$ by integrating $g'(x)$ above.

Method 3.

We use the weighted arithmetic-geometric mean inequality. Given nonnegative real numbers $a_0, \ldots, a_N$ and positive weights $w_0, \ldots, w_N$ with $\sum_n w_n = w$, this states that

$$\frac{1}{w}\sum_{n=0}^{N} w_n a_n \geq \left(\prod_{n=0}^{N} a_n^{w_n}\right)^{\frac{1}{w}}.$$

Given an increasing sequence $x_0, x_1, x_2, \ldots$ with $x_0 = 1$ we set

$$a_n = \frac{3^n x_{n+1}}{x_n^3}, \qquad\qquad w_n = \frac{1}{3^n},$$

and observe that

$$\sum_{n=0}^{N} w_n = \sum_{n=0}^{N} \frac{1}{3^n} = \frac{3^{N+1}-1}{2\cdot 3^N}.$$

Then applying the weighted AM-GM inequality we have

$$\begin{aligned}
\sum_{n=0}^{N} \frac{x_{n+1}}{x_n^3} &= \sum_{n=0}^{N} \frac{1}{3^n} \cdot \frac{3^n x_{n+1}}{x_n^3} \\
&\geq \frac{3^{N+1}-1}{2\cdot 3^N}\left[\prod_{n=0}^{N}\left(\frac{3^n x_{n+1}}{x_n^3}\right)^{\frac{1}{3^n}}\right]^{\frac{2\cdot 3^N}{3^{N+1}-1}} \\
&= \frac{3^{N+1}-1}{2\cdot 3^N}\left[\left(3^{\sum_{n=0}^{N}\frac{n}{3^n}}\right)\left(x_{N+1}^{\frac{1}{3^N}}\right)\right]^{\frac{2\cdot 3^N}{3^{N+1}-1}} \\
&\geq \frac{3^{N+1}-1}{2\cdot 3^N}\left[3^{\sum_{n=0}^{N}\frac{n}{3^n}}\right]^{\frac{2\cdot 3^N}{3^{N+1}-1}}.
\end{aligned} \tag{3}$$

Now taking limits as $N \to \infty$ and using the fact that $\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$ we get

$$\sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3} \geq \frac{3}{2}\left[3^{\frac{3}{4}}\right]^{\frac{2}{3}} = \frac{3\sqrt{3}}{2},$$

as required.

Method 4.

Finally, we outline a calculus solution. Since

$$\sum_{n=0}^{\infty} \frac{x_{n+1}}{x_n^3} \geq \sum_{n=0}^{N} \frac{x_{n+1}}{x_n^3},$$

we will seek to minimise the sum $S_N(\mathbf{x}) = \sum_{n=0}^{N} \frac{x_{n+1}}{x_n^3}$ as a function of $\mathbf{x} = (x_1, \ldots, x_N)$ on $(0, \infty)^N$, treating $x_0 = 1$ and x_{N+1} as constants. In carrying out the minimisation we will ignore the constraint that the sequence is increasing.

Taking the partial derivative of S_N with respect to x_n we have

$$\frac{\partial S_N}{\partial x_n} = \frac{1}{x_{n-1}^3} - \frac{3x_{n+1}}{x_n^4},$$

which is equal to 0 when

$$\frac{x_{n+1}}{x_n^3} = \frac{1}{3} \cdot \frac{x_n}{x_{n-1}^3}. \tag{4}$$

Thus if all partial derivatives are equal to 0 then the terms of the sum are in geometric progression, and

$$S_N = \frac{x_1}{x_0^3} \sum_{n=0}^{N} \frac{1}{3^n} = \frac{3x_1}{2}\left(1 - \frac{1}{3^{N+1}}\right).$$

Equation (4) rearranges to

$$x_{n+1} = \frac{1}{3} \cdot \frac{x_n^4}{x_{n-1}^3},$$

which we may regard as a second order recurrence relation for the values of the x_n at the critical point, with initial conditions $x_0 = 1$ and x_1. The solution can be seen to have the form

$$x_n = \frac{x_1^{a_n}}{3^{b_n}},$$

which leads to a pair of second order linear recurrence relations with constant co-efficients. Solving these we get

$$x_n = \frac{x_1^{(3^n-1)/2}}{3^{(3^n-1-2n)/4}} \tag{5}$$

at the critical point for $1 \leq n \leq N+1$. Recalling now that x_{N+1} is fixed, equation (5) gives

$$x_{N+1} = \frac{x_1^{(3^{N+1}-1)/2}}{3^{(3^{N+1}-3-2N)/4}},$$

which we solve for x_1 to get

$$x_1 = 3^{\frac{3^{N+1}-3-2N}{2(3^{N+1}-1)}} x_{N+1}^{2/(3^{N+1}-1)}.$$

Thus there is a unique critical point $\mathbf{x}^*$, and at this point we have

$$S_N(\mathbf{x}^*) = S_N^* = \frac{3}{2} \cdot 3^{\frac{3^{N+1}-3-2N}{2(3^{N+1}-1)}} x_{N+1}^{2/(3^{N+1}-1)} \left(1 - \frac{1}{3^{N+1}}\right). \tag{6}$$

We claim that $S_N^* = \inf S_N(\mathbf{x})$ on $(0,\infty)^N$. If not, then S_N must take on a smaller value as either $||\mathbf{x}|| \to \infty$ or $x_n \to 0$ for some n. Notice that $S_N^* < C = \frac{3\sqrt{3}}{2} x_{N+1}$. If there exists $0 \le n \le N$ such that $\frac{x_{n+1}}{x_n^3} > C$ then $S_N(\mathbf{x}) > S_N^*$; and if not then we find that

$$x_n \le C^{\frac{3^{n+1}-1}{2}}$$

for $1 \le n \le N$, so the entries of $\mathbf{x}$ are bounded. It follows that $S_N(\mathbf{x}) > S_N^*$ when $||\mathbf{x}||$ is large. In addition, if $\frac{x_{n+1}}{x_n^3} \le C$ for $0 \le n \le N$ then the entries of $\mathbf{x}$ must be bounded away from 0: if $x_n < 1/C$ for some n then $x_{n+1} < 1/C^2 < 1/C$, which inductively gives $x_{N+1} < 1$, a contradiction. We conclude that S_N^* is indeed the minimum of $S_N(\mathbf{x})$ on $(0,\infty)^N$.

To complete the proof that $\inf M = \frac{3\sqrt{3}}{2}$, observe that

$$M \ge S_N^* \ge \frac{3}{2} \cdot 3^{\frac{3^{N+1}-3-2N}{2(3^{N+1}-1)}} \left(1 - \frac{1}{3^{N+1}}\right).$$

Since this holds for all N we may take the limit as $N \to \infty$ to get $M \ge \frac{3\sqrt{3}}{2}$, as required.

Comment. In both Method 3 and 4 we minimised $\sum_{n=0}^{N} \frac{x_{n+1}}{x_n^3}$, while holding x_{N+1} fixed. We note that the respective minima found in equations (3) and (6) are equal. To reconcile the two, use the fact that

$$\begin{aligned}\sum_{n=0}^{N} n x^n &= x \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{N} x^n \\ &= x \frac{\mathrm{d}}{\mathrm{d}x} \frac{x^{N+1}-1}{x-1} \\ &= \frac{N x^{N+2} - (N+1) x^{N+1} + x}{(x-1)^2}.\end{aligned}$$

Substituting $x = \frac{1}{3}$ we obtain

$$\sum_{n=0}^{N} \frac{n}{3^n} = \frac{3^{N+1} - 3 - 2N}{4 \cdot 3^N},$$

and substituting this into equation (3) yields equation (6).

Problem A4

A *regular spatial pentagon* consists of five points P_1, P_2, P_3, P_4, P_5 in $\mathbb{R}^3$ such that $|P_iP_{i+1}| = |P_jP_{j+1}|$ and $\angle P_{i-1}P_iP_{i+1} = \angle P_{j-1}P_jP_{j+1}$ for all $1 \le i, j \le 5$, where $P_0 = P_5$ and $P_6 = P_1$. A regular spatial pentagon is *planar* if there is a plane passing through all five points P_1, P_2, P_3, P_4, P_5.

Show that every regular spatial pentagon is planar.

Solution

We present three solutions. Subscripts should be read modulo 5 throughout.

Solution 1.

Let $\mathcal{P} = P_1P_2P_3P_4P_5$ be a regular spatial pentagon, and consider the triangles $\triangle P_{i-1}P_iP_{i+1}$ and $\triangle P_{j-1}P_jP_{j+1}$ for $1 \le i, j \le 5$. Since $|P_{i-1}P_i| = |P_{j-1}P_j|$, $|P_iP_{i+1}| = |P_jP_{j+1}|$ and $\angle P_{i-1}P_iP_{i+1} = \angle P_{j-1}P_jP_{j+1}$ these triangles are congruent (SAS), so $|P_{i-1}P_{i+1}| = |P_{j-1}P_{j+1}|$. Thus all diagonals of the pentagon are equal in length.

Consider the point P_4. By the previous paragraph this lies on the locus of points P such that $|PP_1| = |PP_2| = |P_1P_3|$, which is a circle C lying in the plane that perpendicularly bisects the line segment P_1P_2. The point P_4 also lies on the sphere S with centre P_3 and radius $|P_2P_3|$. The circle C and sphere S intersect in at most two points, which we label P_4^+ and P_4^- such that P_4^+ lies above the plane of triangle $\triangle P_1P_2P_3$. Exchanging the roles of P_1 and P_3, a similar argument gives us two possible points for P_5, which we similarly label P_5^+ and P_5^-.

If

$$P_4 = P_4^+ \text{ and } P_5 = P_5^+ \qquad \text{or} \qquad P_4 = P_4^- \text{ and } P_5 = P_5^-$$

then reflection in the plane perpendicularly bisecting P_1P_3 is an isometry of $\mathbb{R}^3$ that preserves the pentagon. On the other hand, if

$$P_4 = P_4^+ \text{ and } P_5 = P_5^- \qquad \text{or} \qquad P_4 = P_4^- \text{ and } P_5 = P_5^+$$

then a $180°$ rotation about the angle bisector of $\angle P_1P_2P_3$ is an isometry of $\mathbb{R}^3$ that preserves the pentagon. In either case, there exists an isometry T_1 of $\mathbb{R}^3$ that fixes P_2, swaps P_1 and P_3, and swaps P_4 and P_5.

Applying the same argument to P_4, P_5 and P_1 we get an isometry T_2 of $\mathbb{R}^3$ that fixes P_5, swaps P_1 and P_4, and swaps P_2 and P_3. Then $T = T_2 \circ T_1$ is an isometry of $\mathbb{R}^3$ that sends P_i to P_{i+1} for each i. Let

$$G = \frac{P_1 + P_2 + P_3 + P_4 + P_5}{5}$$

be the centroid of $\mathcal{P}$. Then G is fixed by T, so by taking G as the origin of $\mathbb{R}^3$ we may regard T as an orthogonal linear transformation.

Since 3 is odd T has a real eigenvalue λ and corresponding eigenvector $\mathbf{v}$; moreover $\lambda = \pm 1$ because T is orthogonal. Let π denote the orthogonal projection from $\mathbb{R}^3$ onto the line spanned by $\mathbf{v}$. Since T is orthogonal and $\mathbf{v}$ is an eigenvector of T we have $T \circ \pi = \pi \circ T$, and so

$$\pi(P_{i+1}) = \pi(T(P_i)) = T(\pi(P_i)) = \lambda \pi(P_i) \tag{7}$$

for $1 \leq i \leq 5$.

Applying equation (7) repeatedly we get $\pi(P_i) = \lambda^{i-1}\pi(P_1)$ for $1 \leq i \leq 6$. In particular $\pi(P_1) = \lambda^5 \pi(P_1)$, so either $\pi(P_1) = \mathbf{0}$ or $\lambda = 1$. In either case equation (7) gives $\pi(P_i) = \pi(P_j)$ for $1 \leq i, j \leq 5$, so the P_i all lie on a plane orthogonal to $\mathbf{v}$. This shows that $\mathcal{P}$ is planar.

Comment. When $\lambda = 1$ we still get $\pi(P_1) = \mathbf{0}$, because then

$$\begin{aligned}
\pi(P_1) &= \frac{\pi(P_1) + \pi(P_2) + \pi(P_3) + \pi(P_4) + \pi(P_5)}{5} \\
&= \pi\left(\frac{P_1 + P_2 + P_3 + P_4 + P_5}{5}\right) \\
&= \pi(G) = \mathbf{0}.
\end{aligned}$$

Solution 2.

As shown in Solution 1, all diagonals $|P_iP_{i+2}|$ are equal in length. In fact, the conditions of the problem can be equivalently rephrased as $|P_iP_{i+1}|$ are equal for all i, and $|P_iP_{i+2}|$ are equal for all i. A suitable relabelling (say $Q_i = P_{2i}$) would allow us to assume without loss of generality that $|P_iP_{i+1}| \leq |P_iP_{i+2}|$.

Denote the midpoint of P_iP_j by $M_{i,j}$. Now we make the following claim:

> If $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ are not coplanar, then $P_{i-1}, M_{i,i+3}, M_{i+1,i+2}$ are collinear. Moreover, this line is perpendicular to both P_iP_{i+3} and $P_{i+1}P_{i+2}$.

Proof. First, $|M_{i,i+3}P_{i+1}| = |M_{i,i+3}P_{i+2}|$ since they are corresponding medians of congruent triangles $P_iP_{i+3}P_{i+1}$ and $P_{i+3}P_iP_{i+2}$. Also recall that $|P_{i-1}P_{i+1}| = |P_{i-1}P_{i+2}|$. Hence $M_{i,i+3}, M_{i+1,i+2}$ and P_{i-1} all lie on the plane perpendicularly bisecting $P_{i+1}P_{i+2}$. By similar arguments (or via the aforementioned relabelling), $M_{i,i+3}, M_{i+1,i+2}$ and P_{i-1} also all lie on the plane perpendicularly bisecting P_iP_{i+3}. Since $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ are not planar, the line segments $P_{i+1}P_{i+2}$, P_iP_{i+3} are skew, so their perpendicular bisecting planes are distinct. Therefore $M_{i,i+3}, M_{i+1,i+2}$ and P_{i-1} must lie on the intersection of the two distinct planes, which is a line perpendicular to both P_iP_{i+3} and $P_{i+1}P_{i+2}$. □

Returning to the problem, suppose that the five points are not coplanar. Without loss of generality suppose that P_1, P_2, P_3, P_4 are not planar; then by the claim above $M_{1,4}, M_{2,3}$

and P_5 all lie on a line ℓ. Denote the plane containing $P_1P_4P_5$ by ϕ, and note that ℓ lies on ϕ because both P_5 and $M_{1,4}$ do. Since P_2P_3 meets ℓ perpendicularly at its midpoint $M_{2,3}$, neither P_2 nor P_3 can lie on ϕ, because otherwise both would and our pentagon would be planar. Moreover P_2 and P_3 must lie on opposite sides of ϕ. Thus we can apply our claim again to say $\{P_3, P_4, P_5, P_1\}$, to conclude that $M_{1,3}, M_{4,5}, P_2$ are collinear. Since $M_{1,3}$ and P_2 are on opposite sides of ϕ and $M_{4,5}$ is on ϕ, the point $M_{4,5}$ must be between $M_{1,3}$ and P_2.

We now project P_2, P_3 and $M_{1,3}$ onto ϕ to obtain P_2', P_3' and $M_{1,3}'$. Note that all other points used so far already lie on ϕ. Then $P_2'P_3'P_4P_1$ is an isosceles trapezium (in that order since $P_1P_2 \leq P_1P_3$), with ℓ as the line of symmetry. Moreover, $P_2'P_3' < P_2P_3 \leq P_4P_1$. Therefore $M_{1,3}'$ is on the same side of ℓ as P_1 and P_2', which are on the opposite side of ℓ to $M_{4,5}$. Therefore $M_{4,5}$ cannot lie in between $M_{1,3}'$ and P_2', which is a contradiction.

Solution 3.

Let $\mathcal{P} = P_1P_2P_3P_4P_5$ be a regular spatial pentagon, and for each i let $\mathbf{v}_i$ be the vector from P_i to P_{i+1}. Without loss of generality we may assume that $|\mathbf{v}_i| = 1$ for all i.

Since $\theta = \angle P_{i-1}P_iP_{i+1}$ is constant for all i, we have

$$\mathbf{v}_i \cdot \mathbf{v}_{i+1} = |\mathbf{v}_i||\mathbf{v}_{i+1}| \cos(\pi - \theta) = -\cos(\theta),$$

so $\mathbf{v}_i \cdot \mathbf{v}_{i+1}$ is also constant. Note also that $\sum_{i=1}^5 \mathbf{v}_i = \mathbf{0}$. Therefore

$$\mathbf{v}_j \cdot \sum_{i=1}^5 \mathbf{v}_i = 0 = \mathbf{v}_{j-2} \cdot \sum_{i=1}^5 \mathbf{v}_i,$$

and expanding and cancelling common terms we find that

$$\begin{aligned} \mathbf{v}_j \cdot \mathbf{v}_{j+2} &= \mathbf{v}_{j-2} \cdot \mathbf{v}_{j+1} \\ &= \mathbf{v}_{j+1} \cdot \mathbf{v}_{j+3} && \text{(using } \mathbf{v}_{j-2} = \mathbf{v}_{j+3}\text{)}. \end{aligned}$$

It follows that $\mathbf{v}_i \cdot \mathbf{v}_{i+2}$ is constant. So there are constants a and b such that for all indices i we have

$$\mathbf{v}_i \cdot \mathbf{v}_i = 1, \qquad \mathbf{v}_i \cdot \mathbf{v}_{i+1} = a, \qquad \mathbf{v}_i \cdot \mathbf{v}_{i+2} = b, \qquad \mathbf{v}_i \cdot \mathbf{v}_{i+3} = b, \qquad \mathbf{v}_i \cdot \mathbf{v}_{i+4} = a.$$

Let A be the 5×5 matrix whose (i, j)-entry is $\mathbf{v}_i \cdot \mathbf{v}_j$, so that

$$A = \begin{bmatrix} 1 & a & b & b & a \\ a & 1 & a & b & b \\ b & a & 1 & a & b \\ b & b & a & 1 & a \\ a & b & b & a & 1 \end{bmatrix}.$$

Notice that each row of A is obtained from the one above by rotating the entries one step to the right. Matrices with this property are called *circulant matrices*, and in the 5×5 case their eigenvectors are given by

$$\mathbf{w}_j = \begin{bmatrix} 1 & \omega^j & \omega^{2j} & \omega^{3j} & \omega^{4j} \end{bmatrix}^T$$

for $0 \leq j \leq 4$, where $\omega = e^{2\pi i/5}$ is a primitive 5th root of unity. The corresponding eigenvalues are

$$\lambda_j = 1 + a\omega^j + b\omega^{2j} + b\omega^{3j} + a\omega^{4j},$$

as is easily checked by multiplication.

We will use the eigenvalues and eigenvectors to show that the null space of A is 3-dimensional. First note that the $\mathbf{w}_j$ are linearly independent, so the dimension of the null space is equal to the number of 0 eigenvalues. Next, observe that $\lambda_1 = \lambda_4$ and $\lambda_2 = \lambda_3$; and note also that $\lambda_0 = 0$, because

$$\begin{aligned}
\lambda_0 &= 1 + a + b + b + a \\
&= \sum_{i=1}^{5} \mathbf{v}_1 \cdot \mathbf{v}_i \\
&= \mathbf{v}_1 \cdot \sum_{i=1}^{5} \mathbf{v}_i \\
&= \mathbf{v}_1 \cdot \mathbf{0} = 0.
\end{aligned}$$

Combining these facts we see that the null space of A has dimension 1, 3 or 5. However, A cannot have a 5-dimensional null space, because it has rank at least 1, so the null space must have dimension 1 or 3. We will show that the null space is 3-dimensional by exhibiting two linearly independent vectors in the null space.

Since the $\mathbf{v}_i$ lie in $\mathbb{R}^3$, any four of them must be a linearly dependent set. Let c_1, c_2, c_3, c_4 be constants not all equal to 0 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = 0$, and let $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & 0 \end{bmatrix}^T$. Then for $1 \leq i \leq 5$ the ith entry of $A\mathbf{c}$ is given by

$$\begin{aligned}
(A\mathbf{c})_i &= c_1(\mathbf{v}_i \cdot \mathbf{v}_1) + c_2(\mathbf{v}_i \cdot \mathbf{v}_2) + c_3(\mathbf{v}_i \cdot \mathbf{v}_3) + c_4(\mathbf{v}_i \cdot \mathbf{v}_4) \\
&= \mathbf{v}_i \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4) \\
&= \mathbf{v}_i \cdot \mathbf{0} = 0,
\end{aligned}$$

so $\mathbf{c}$ belongs to $NS(A)$. Since $\mathbf{w}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$ also belongs to the null space, and $\mathbf{w}_0$ and $\mathbf{c}$ are linearly independent, we conclude that $\dim NS(A) \geq 2$ and hence $\dim NS(A) = 3$.

Write $\mathbf{a}_i$ for the ith column of A. Since $\text{rank}(A) = 5 - \dim NS(A) = 5 - 3 = 2$, given $1 \leq x < y < z \leq 5$ there exist constants b_x, b_y, b_z not all equal to 0 such that

$$b_x\mathbf{a}_x + b_y\mathbf{a}_y + b_z\mathbf{a}_z = \mathbf{0}.$$

Then for $1 \leq i \leq 5$ we have

$$\mathbf{v}_i \cdot (b_x \mathbf{v}_x + b_y \mathbf{v}_y + b_z \mathbf{v}_z) = b_x(\mathbf{v}_i \cdot \mathbf{v}_x) + b_y(\mathbf{v}_i \cdot \mathbf{v}_y) + b_z(\mathbf{v}_i \cdot \mathbf{v}_z) = 0,$$

so the projection of $\mathbf{u} = b_x \mathbf{v}_x + b_y \mathbf{v}_y + b_z \mathbf{v}_z$ onto $\mathbf{v}_i$ has length 0 for all i. Since $\mathbf{u}$ is a linear combination of the $\mathbf{v}_i$ it must therefore be the zero vector, and it follows that $\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z$ are linearly dependent. We conclude that the $\mathbf{v}_i$ span a 2-dimensional subspace, showing that our spatial pentagon $\mathcal{P}$ is planar.

Problem B1

Let $\mathcal{M}$ be the set of 5×5 real matrices of rank 3. Given a matrix A in $\mathcal{M}$, the set of columns of A has $2^5 - 1 = 31$ nonempty subsets. Let k_A be the number of these subsets that are linearly independent.

Determine the maximum and minimum values of k_A, as A varies over $\mathcal{M}$.

The rank *of a matrix is the dimension of the span of its columns.*

Solution

The maximum possible value of k_A is 25, and the minimum possible value is 7.

Let $A \in \mathcal{M}$. Then since $\operatorname{rank}(A) = 3$ a linearly independent subset of the set of columns of A has size at most 3, so

$$k_A \leq \binom{5}{1} + \binom{5}{2} + \binom{5}{3} = 5 + 10 + 10 = 25.$$

Now let S be a linearly independent subset of the columns of A of size 3. Then every nonempty subset of S is also linearly independent, so

$$k_A \geq 2^3 - 1 = 7.$$

We conclude that $7 \leq k_A \leq 25$.

To complete the proof let

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $A_1, A_2 \in \mathcal{M}$ and we have $k_{A_1} = 7$, $k_{A_2} = 25$, showing that these bounds are sharp.

Problem B2

For each positive integer k, let S_k be the set of real numbers that can be expressed in the form

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k},$$

where $n_1, n_2, \ldots, n_k$ are positive integers.

Prove that S_k does not contain an infinite strictly increasing sequence.

Solution

Solution 1.

We will prove the statement by induction on k. Observe that the statement is clearly true when $k = 1$.

Now consider an integer $k \geq 2$, and assume that S_{k-1} does not contain an infinite strictly increasing sequence. In order to obtain a contradiction, suppose that S_k contains an infinite strictly increasing sequence $\{x_i\}_{i \in \mathbb{N}}$. Write

$$x_i = \frac{1}{n_{i,1}} + \cdots + \frac{1}{n_{i,k}}$$

where $n_{i,j} \leq n_{i,j+1}$ for all $i \in \mathbb{N}$ and $1 \leq j < k$. Choose $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < x_1$. Then

$$\varepsilon < x_i = \frac{1}{n_{i,1}} + \frac{1}{n_{i,2}} + \cdots + \frac{1}{n_{i,k}} \leq \frac{k}{n_{i,1}},$$

which implies

$$n_{i,1} \leq \frac{k}{\varepsilon}$$

for all i. So there are only a finite number of possibilities for $n_{i,1}$. It follows that infinitely many of the sequences $n_{i,1}, n_{i,2}, \ldots, n_{i,k}$ must share the same first term m. Let these sequences be $m = m_{i,1} \leq m_{i,2} \leq m_{i,3} \leq \cdots \leq m_{i,k}$, where $i = 1, 2, 3, \ldots$. Then we have

$$\sum_{j=2}^{k} \frac{1}{m_{1,j}} < \sum_{j=2}^{k} \frac{1}{m_{2,j}} < \sum_{j=2}^{k} \frac{1}{m_{3,j}} < \cdots.$$

However, this contradicts the inductive hypothesis. Therefore, S_k does not contain an infinite strictly increasing sequence and the induction is complete.

Solution 2.

Using the same set-up and notation as Solution 1 we give an alternative proof of the inductive step.

As above suppose that S_k contains an infinite strictly increasing sequence $\{x_i\}_{i \in \mathbb{N}}$, where

$$x_i = \frac{1}{n_{i,1}} + \cdots + \frac{1}{n_{i,k}}$$

with $n_{i,j} \le n_{i,j+1}$ for all $1 \le j < k$. We claim that the sequence $\{n_{i,k}\}_{i\in\mathbb{N}}$ is unbounded. In order to see this, assume it is instead bounded above by some M. Then there are only finitely many possibilities for $n_{i,k}$, and by a similar argument to Solution 1 we get an infinite strictly increasing sequence in S_{k-1}, contrary to the inductive hypothesis. It therefore follows that there exists an increasing subsequence $\{n_{\phi(i),k}\}_{i\in\mathbb{N}}$; furthermore, $\{\frac{1}{n_{\phi(i),k}}\}_{i\in\mathbb{N}}$ is a decreasing sequence. We can then see that for all $j > i$,

$$x_{\phi(i)} - \frac{1}{n_{\phi(i),k}} < x_{\phi(j)} - \frac{1}{n_{\phi(j),k}}.$$

But $\{x_{\phi(i)} - \frac{1}{n_{\phi(i),k}}\}_{i\in\mathbb{N}}$ is also an element in S_{k-1}. So, if an infinite strictly increasing sequence exists in S_k, then one also exists in S_{k-1}. Taking the contrapositive proves our inductive hypothesis.

Solution 3.

Let T_k be the set of all k-tuples of positive integers; that is,

$$T_k = \left\{(n_1, n_2, \ldots, n_k) : n_i \in \mathbb{Z}^+\right\}.$$

We first prove the following lemma:

Lemma. *Let $\{a_i\}_{i=1}^{\infty}$ be an infinite sequence in T_k, where $a_i = (n_{i,1}, n_{i,2}, \ldots, n_{i,k})$. Then there exists a subsequence $\{a_{i_t}\}_{t=1}^{\infty}$ such that $\{n_{i_t,j}\}_{t=1}^{\infty}$ is a nondecreasing sequence for each $1 \le j \le k$.*

Proof. We construct the subsequence inductively, working with one component at a time. By repeatedly passing to a subsequence, it's sufficient to show that we can construct a subsequence $\{a_{i_t}\}_{t=1}^{\infty}$ such that $\{n_{i_t,j}\}_{t=1}^{\infty}$ is a nondecreasing sequence for some fixed j such that $1 \le j \le k$.

Consider the sequence $\{n_{i,j}\}_{i=1}^{\infty}$. Since $n_{i,j}$ is a positive integer for each i, we have two cases:

1. *Some integer m appears in the sequence $\{n_{i,j}\}_{i=1}^{\infty}$ infinitely often.* In this case, we can take $\{a_{i_t}\}_{t=1}^{\infty}$ to be the subsequence of the a_i such that the jth entry is equal to m.
2. *No integer m appears in the sequence $\{n_{i,j}\}_{i=1}^{\infty}$ infinitely often.* In this case, the values of the sequence $\{n_{i,j}\}_{i=1}^{\infty}$ must be unbounded, so there exists a strictly increasing subsequence $\{n_{i_t,j}\}_{t=1}^{\infty}$. We take $\{a_{i_t}\}_{t=1}^{\infty}$ to be the corresponding subsequence of the a_i.

In both cases, we are able to construct an infinite subsequence $\{a_{i_t}\}_{t=1}^{\infty}$ such that the sequence $\{n_{i_t,j}\}_{t=1}^{\infty}$ is nondecreasing. We now work with this subsequence only, and repeat the argument above to find a subsequence of $\{a_{i_t}\}_{t=1}^{\infty}$ such that the $(j+1)$th entries are nondecreasing. After carrying out this process a total of k times we will arrive at the required subsequence. □

Returning to the original problem, let $\{x_i\}_{i=1}^{\infty}$ be an infinite sequence in S_k, and for each i let

$$a_i = (n_{i,1}, \ldots, n_{i,k}) \in T_k$$

be such that

$$x_i = \frac{1}{n_{i,1}} + \cdots + \frac{1}{n_{i,k}}.$$

By the lemma there exists a subsequence $\{a_{i_t}\}_{t=1}^{\infty}$ such that $\{n_{i_t,j}\}_{t=1}^{\infty}$ is a nondecreasing sequence for each $1 \leq j \leq k$. Then $\{1/n_{i_t,j}\}_{t=1}^{\infty}$ is nonincreasing for each $1 \leq j \leq k$, and since

$$x_{i_t} = \frac{1}{n_{i_t,1}} + \cdots + \frac{1}{n_{i_t,k}}$$

it follows that the subsequence $\{x_{i_t}\}_{t=1}^{\infty}$ is too. Therefore any infinite sequence $\{x_i\}_{i=1}^{\infty}$ of S_k contains a nonincreasing subsequence, so it cannot be strictly increasing. This completes the proof.

Solution 4.

First note that the set S_k is bounded above by k, because each term in the sum

$$\frac{1}{n_1} + \cdots + \frac{1}{n_k}$$

is bounded above by 1. This means that any infinite strictly increasing sequence in S_k must converge to some limit. To show that no such sequence exists, we prove for each $L > 0$ that there is no infinite strictly increasing sequence in S_k that converges to L. We do this by proving the following statement:

> For each $L \in \mathbb{R}$ there exists $\varepsilon_L^k > 0$ such that no element of S_k lies in the interval $(L - \varepsilon_L^k, L)$.

This implies that there is no sequence in S_k that converges to L from below.

We prove the claim by induction on k.

Base Case: Let $k = 1$. If $L > 1$ then we may set $\varepsilon_L^1 = L - 1 > 0$; and if $L \leq 0$ then we may set $\varepsilon_L^1 = 1$. Otherwise, for each $0 < L \leq 1$, there exists an integer m such that $\frac{1}{m} < L \leq \frac{1}{m-1}$. All elements of S_1 are of the form $\frac{1}{n}$ for $n \in \mathbb{N}$, so there are no elements of S_1 inside the interval $(\frac{1}{m}, L)$. We may therefore take $\varepsilon_L^1 = L - \frac{1}{m} > 0$, so the required real number ε_L^1 exists.

Inductive Step: Now suppose that the statement holds for S_k for some integer k; that is, assume that for each $L \in \mathbb{R}$, there exists $\varepsilon_L^k > 0$ such that no element of S_k lies in the interval $(L - \varepsilon_L^k, L)$. Fix $L \in \mathbb{R}$. We show that there exists $\varepsilon_L^{k+1} > 0$ such that no element of S_{k+1} lies in the interval $(L - \varepsilon_L^{k+1}, L)$.

To do this, first note that each element in S_{k+1} can be expressed as $a + \frac{1}{p}$, where a is an element of S_k and p is a positive integer. For each $p \in \mathbb{N}$ define

$$A_p = \left\{ a + \frac{1}{p} : a \in S_k \right\};$$

then

$$S_{k+1} = \bigcup_{p \in \mathbb{N}} A_p.$$

Let $\varepsilon_L^k > 0$ be such that no element of S_k belongs to the interval $(L - \varepsilon_L^k, L)$, and let $m \in \mathbb{N}$ be such that $\varepsilon_L^k > \frac{1}{m}$. We consider two cases.

1. $p \geq m$. By choice of ε_L^k no element of S_k lies in the interval $(L - \varepsilon_L^k, L)$. Adding $\frac{1}{p}$ to each element of S_k, it follows that no element of A_p lies in the interval $(L - \varepsilon_L^k + \frac{1}{p}, L)$. Set
$$\hat{\varepsilon}_L^{k+1} = \varepsilon_L^k - \frac{1}{m} > 0.$$
Then
$$\hat{\varepsilon}_L^{k+1} = \varepsilon_L^k - \frac{1}{m} < \varepsilon_L^k - \frac{1}{p},$$
so for all $p \geq m$ no element of A_p lies in the interval $(L - \hat{\varepsilon}_L^{k+1}, L)$.

2. $p < m$. Consider $L - \frac{1}{p}$. From the inductive assumption, there exists $\varepsilon_{L-\frac{1}{p}}^k > 0$ such that no element of S_k lies in the interval $(L - \frac{1}{p} - \varepsilon_{L-\frac{1}{p}}^k, L - \frac{1}{p})$. Adding $\frac{1}{p}$ to each element of S_k, it follows that no element of A_p lies in the interval $(L - \varepsilon_{L-\frac{1}{p}}^k, L)$.

   Since there are only finitely many p such that $p < m$ we may set
$$\tilde{\varepsilon}_L^{k+1} = \min_{p<m} \left\{ \varepsilon_{L-\frac{1}{p}}^k \right\} > 0.$$
   Then for all $p < m$ no element of A_p lies in the interval $(L - \tilde{\varepsilon}_L^{k+1}, L)$.

Set

$$\varepsilon_L^{k+1} = \min\left(\hat{\varepsilon}_L^{k+1}, \tilde{\varepsilon}_L^{k+1}\right) > 0.$$

Then from our work above the interval $(L - \varepsilon_L^{k+1}, L)$ contains no element of A_p for any $p \in \mathbb{N}$, and so contains no element of S_{k+1}. This establishes the inductive step and completes the proof.

Problem B3

A cat is trying to catch a mouse in the nonnegative quadrant

$$N = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}.$$

At time $t = 0$ the cat is at $(1, 1)$ and the mouse is at $(0, 0)$. The cat moves with speed $\sqrt{2}$ such that its position $c(t) = (c_1(t), c_2(t))$ is continuous, and differentiable except at finitely many points; while the mouse moves with speed 1 such that its position $m(t) = (m_1(t), m_2(t))$ is also continuous, and differentiable except at finitely many points. Thus

$$c(0) = (1, 1), \qquad\qquad m(0) = (0, 0);$$

$c(t)$ and $m(t)$ are continuous functions of t such that $c(t), m(t) \in N$ for all $t \geq 0$; the derivatives $c'(t) = (c_1'(t), c_2'(t))$ and $m'(t) = (m_1'(t), m_2'(t))$ each exist for all but finitely many t; and

$$(c_1'(t))^2 + (c_2'(t))^2 = 2, \qquad\qquad (m_1'(t))^2 + (m_2'(t))^2 = 1,$$

whenever the respective derivative exists.

At each time t the cat knows both the mouse's position $m(t)$ and velocity $m'(t)$. Show that, no matter how the mouse moves, the cat can catch it by time $t = 1$; that is, show that the cat can move such that $c(\tau) = m(\tau)$ for some $\tau \in [0, 1]$.

Solution

We may treat the function m as given. We will show that the cat can catch the mouse by showing that the derivative c' may be chosen so as to guarantee that $c(\tau) = m(\tau)$ for some $\tau \in [0, 1]$, with the value of $c'(t)$ depending only on the values of $m(t)$ and $m'(t)$ for each t.

Let $t^* \in [0, 1]$ be the least t such that

$$m_1(t) = 1 - t \qquad \text{or} \qquad m_2(t) = 1 - t. \tag{8}$$

Such a t necessarily exists, as we now show. For each $i = 1, 2$ the function z_i defined by $z_i(t) = m_i(t) - t + 1$ is continuous and satisfies $z_i(0) = -1$, $z_i(1) = m_1(1) \geq 0$. It therefore follows from the intermediate value theorem that the set

$$Z_i = [0, 1] \cap z_i^{-1}(0)$$

is nonempty, and moreover it is compact because $z_i^{-1}(0)$ is the preimage of a compact set under a continuous function. Then $Z = Z_1 \cup Z_2$ is nonempty and compact also, and so contains its infimum. If $t \in [0, 1]$ then t satisfies (8) if and only if $t \in Z$, so $t^* = \inf Z$ exists, as claimed.

For $0 \le t \le t^*$ we define c' by $c'(t) = (-1,-1)$, so that $c(t) = (1-t, 1-t)$. Observe that $(c_1'(t))^2 + (c_2'(t))^2 = 2$, as required. At $t = t^*$ we have either $c_1(t^*) = m_1(t^*)$ or $c_2(t^*) = m_2(t^*)$; without loss of generality we may assume that $c_1(t^*) = m_1(t^*)$. If also $c_2(t^*) = m_2(t^*)$ holds then the cat has caught the mouse, and we are done; note also that this necessarily holds if $t^* = 1$. So we will assume that $t^* < 1$ and $c_2(t^*) \neq m_2(t^*)$. This implies $c_2(t^*) > m_2(t^*)$.

Now for $t^* \le t \le 1$ we define c' by

$$c_1'(t) = m_1'(t), \qquad\qquad c_2'(t) = -\sqrt{2-(m_1'(t))^2},$$

so that

$$\begin{aligned} c_1(t) &= c_1(t^*) + \int_{t^*}^{t} m_1'(s)\,\mathrm{d}s = m_1(t), \\ c_2(t) &= c_2(t^*) - \int_{t^*}^{t} \sqrt{2-(m_1'(s))^2}\,\mathrm{d}s. \end{aligned}$$

Observe that c is continuous, and differentiable except possibly at $t = t^*$ and the at most finitely many points where m is not differentiable. Since $c_1(t) = m_1(t)$ for all $t \ge t^*$ and $c_2(t^*) > m_2(t^*)$, to prove that the cat catches the mouse it suffices to show that the cat reaches the x-axis at some time $t^\dagger \in [t^*, 1]$.

Indeed, since $\sqrt{2-(m_1'(s))^2} \ge 1$ we have

$$\begin{aligned} c_2(1) &= c_2(t^*) - \int_{t^*}^{1} \sqrt{2-(m_1'(s))^2}\,\mathrm{d}s \\ &\le c_2(t^*) - \int_{t^*}^{1} 1\,\mathrm{d}s \\ &= (1-t^*) - (1-t^*) = 0. \end{aligned}$$

Since $m_2(t) \ge 0$ for all t, by the intermediate value theorem there exists $\tau \in [t^*, 1]$ such that $c_2(\tau) = m_2(\tau)$. In view of the fact that $c_1(t) = m_1(t)$ for all $t \ge t^*$ it follows that $c(\tau) = m(\tau)$, so the cat has caught the mouse at time $t = \tau$ and we are done.

Problem B4

The following problem is open in the sense that no solution is currently known to part (b). A proof of part (a) will be awarded 3 points.

Let $n \geq 2$ be an integer, and let P_n be a regular polygon with $n^2 - n + 1$ vertices. We say that n is *taut* if it is possible to choose n of the vertices of P_n such that the pairwise distances between the chosen vertices are all distinct.

(a) Show that if $n - 1$ is prime then n is taut.

(b) Which integers $n \geq 2$ are taut?

Set-up

Let $N = n^2 - n + 1$, and label the vertices of P_N consecutively using the elements of $\mathbb{Z}_N$. Then the distance between vertices i and j is equal to the distance between vertices k and ℓ if and only if $i - j = \pm(k - \ell)$ in $\mathbb{Z}_N$. Our goal then is to choose an n element subset D of $\mathbb{Z}_N$ such that the $n^2 - n$ pairwise differences $i - j$ are distinct for all $(i, j) \in D^2$ with $i \neq j$. Since there are exactly $n^2 - n$ possible nonzero differences in $\mathbb{Z}_N$, each possible difference must occur exactly once.

Solution to part (a)

Suppose that $n - 1 = p$ is prime, and observe that $N = n^2 - n + 1 = p^2 + p + 1 = \frac{p^3-1}{p-1}$. Let $F \cong \mathbb{Z}_p$ be the field of order p. Our construction is motivated by the fact that $N = p^2 + p + 1$ is the number of points in $FP^2 = (F^3 - \{0\})/F^*$, the projective plane over F, and $n = p + 1$ is the number of points in the projective line $FP^1 = (F^2 - \{0\})/F^*$. The key step is to first identify FP^2 with the cyclic group of order N.

We will use the following facts about finite fields, taught in some advanced undergraduate algebra courses:

1. There exists an extension field E of F of order p^3.

2. The multiplicative group of a finite field is cyclic.

3. An element of E that does not belong to F is not the root of any nonzero polynomial over F of degree 2 or less.

Let $G = E^*$, $H = F^*$ be the multiplicative groups of E and F respectively, and let $K = G/H$. Since G and H are cyclic so is K, and since

$$|K| = \frac{|G|}{|H|} = \frac{p^3 - 1}{p - 1} = N$$

we have $K \cong \mathbb{Z}_N$. We may regard E as a 3-dimensional vector space over F, and then as sets we have

$$K = E^*/F^* = (F^3 - \{0\})/F^* = FP^2.$$

Thus we have succeeded in identifying $\mathbb{Z}_N$ with the projective plane over F. From this viewpoint, cosets of H in G correspond to (the nonzero points of) 1-dimensional subspaces of E.

We will prove that n is taut by exhibiting an n-element subset D of K such that the n^2-n pairwise quotients x/y are distinct for all $(x, y) \in D^2$ with $x \neq y$. Following the "wishful thinking" strategy that motivated us to consider FP^2, we will seek to show that the $p+1$ points belonging to a projective line have the required property. To this end choose any $\alpha \in E - F$. By fact 3 above $\{1, \alpha, \alpha^2\}$ is a linearly independent set, so $\operatorname{span}\{1, \alpha\}$ is a 2-dimensional subspace of E. Let

$$\begin{aligned}\tilde{D} &= \operatorname{span}\{1, \alpha\} - \{0\} \\ &= \{s + t\alpha : (s, t) \in F^2, (s, t) \neq (0, 0)\} \subseteq G.\end{aligned}$$

Then $\tilde{D}$ is closed under the action of H by multiplication, and so is a union of cosets of H. We let

$$D = \tilde{D}/H = \{xH : x \in \tilde{D}\} \subseteq K,$$

and claim that D has the required properties.

To prove this note first that each element of D may be written uniquely as xH for $x \in \hat{D} = \{1\} \cup \{t + \alpha : t \in F\} \subseteq G$, because any two elements of $\hat{D}$ are linearly independent over F and $F^*\hat{D} = \tilde{D}$. This confirms that $|D| = p + 1 = n$. Suppose now that $x_1, x_2, x_3, x_4 \in \hat{D}$ satisfy $(x_1/x_2)H = (x_3/x_4)H$ with $x_1 \neq x_2$ and $x_3 \neq x_4$. Let

$$x_i = \varepsilon_i \alpha + t_i$$

for each i, where $\varepsilon_i \in \{0, 1\}$, $t_i \in F$, and $t_i = 1$ if $\varepsilon_i = 0$. Then

$$\frac{\varepsilon_1\alpha + t_1}{\varepsilon_2\alpha + t_2} = \frac{\varepsilon_3\alpha + t_3}{\varepsilon_4\alpha + t_4} h$$

for some $h \in H = F^*$. Multiplying through we get

$$\varepsilon_1\varepsilon_4\alpha^2 + (\varepsilon_1 t_4 + \varepsilon_4 t_1)\alpha + t_1 t_4 = h(\varepsilon_2\varepsilon_3\alpha^2 + (\varepsilon_2 t_3 + \varepsilon_3 t_2)\alpha + t_2 t_4).$$

Since $\{1, \alpha, \alpha^2\}$ is linearly independent over F this means that

$$\begin{aligned}\varepsilon_1\varepsilon_4 &= h\varepsilon_2\varepsilon_3 \\ \varepsilon_1 t_4 + \varepsilon_4 t_1 &= h(\varepsilon_2 t_3 + \varepsilon_3 t_2) \\ t_1 t_4 &= h t_2 t_3.\end{aligned}$$

We consider two cases, according to whether or not $\varepsilon_1\varepsilon_4 = 0$.

If $\varepsilon_1\varepsilon_4 \neq 0$ then we have $\varepsilon_1\varepsilon_4 = h\varepsilon_2\varepsilon_3 = 1$. This implies $\varepsilon_i = 1$ for all i, and so $h = 1$ also. Our system of equations becomes

$$t_1 + t_4 = t_2 + t_3,$$
$$t_1t_4 = t_2t_3.$$

Let $t_1 + t_4 = t_2 + t_3 = s$. Then $t_4 = s - t_1$, $t_3 = s - t_2$, and substituting in $t_1t_4 = t_2t_3$ we get $t_1(s - t_1) = t_2(s - t_2)$. This rearranges to

$$s(t_1 - t_2) = t_1^2 - t_2^2 = (t_1 + t_2)(t_1 - t_2),$$

and since $t_1 \neq t_2$ by our assumption $x_1 \neq x_2$ it follows that $t_1 + t_2 = s$. So

$$s = t_1 + t_2 = t_1 + t_4 = t_2 + t_3,$$

which implies $t_1 = t_3$ and $t_2 = t_4$. We conclude that $x_1 = x_3$ and $x_2 = x_4$.

If $\varepsilon_1\varepsilon_4 = 0$ then we may assume without loss of generality that $\varepsilon_1 = 0$. Then $t_1 = 1$, and $\varepsilon_2 \neq 0$ by our assumption that $x_1 \neq x_2$. So $\varepsilon_2 = 1$ and our system of equations becomes

$$\begin{aligned} \varepsilon_3 &= 0 \\ h(t_3 + \varepsilon_3 t_2) &= \varepsilon_4 \\ ht_2t_3 &= t_4. \end{aligned}$$

From the first equation $t_3 = 1$, and $\varepsilon_4 \neq 0$ by our assumption $x_3 \neq x_4$. So $\varepsilon_4 = 1$ and then $h = h(t_3 + \varepsilon_3 t_2) = \varepsilon_4 = 1$, $t_4 = ht_2t_3 = t_2$. We conclude that $x_1 = x_3 = 1$ and $x_2 = x_4 = 1 + t_2$. This completes the proof that D has the required properties.

Discussion

As discussed above under "Set-up", letting $N = n^2 - n + 1$ the problem is equivalent to determining for which n the cyclic group $\mathbb{Z}_N$ contains an n element subset D such that each nonzero element g of $\mathbb{Z}_N$ can be written uniquely as $g = i - j$ for $i, j \in D$. More generally, let G be an abelian group and let D be a subset of G. Then D is an *abelian planar difference set* if every nontrivial element $g \in G$ can be written uniquely as $g = x - y$ for $x, y \in D$. In these terms, the problem asks for which n does $\mathbb{Z}_N$ contain an abelian planar difference set. (The term "taut" used here does not appear elsewhere in the mathematical literature.)

The argument used above to answer part (a) goes through unchanged if p is replaced by any prime power p^k. It follows that n is taut whenever $n-1$ is a prime power, and so also an abelian planar difference set of cardinality n exists whenever $n - 1$ is a prime power. This was first proved by Singer [2] in 1938.

The *Prime Power Conjecture* (PPC) is the converse statement that an abelian planar difference set of cardinality n exists only if $n-1$ is a prime power. This would imply that n is taut if and only if $n - 1$ is a prime power. Gordon [1] has verified the PPC for n up to two million.

References

[1] Daniel M. Gordon. The prime power conjecture is true for $n < 2,000,000$. *Electron. J. Combin.*, 1:Research Paper 6, approx. 7, 1994.

[2] James Singer. A theorem in finite projective geometry and some applications to number theory. *Trans. Amer. Math. Soc.*, 43(3):377–385, 1938.

