SIMON MARAIS MATHEMATICS COMPETITION

2021 SESSION B: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem B1–B3 is worth 7 points. Points for problem B4 are as given.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions. Tablets may be used solely for writing worked solutions, with internet access switched off.
- For participants sitting the exam off-site, an electronic device such as a PC, laptop, phone or tablet may be used during the competition for accessing the papers, undergoing invigilation, writing and submitting solutions and (for pairs entrants) communicating with the other member of the pair.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "B3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and session (as in **S1234567_B** (for singles) or **P3141593_B** (for pairs), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

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2021 SESSION B: PROBLEMS

B1. Let $n \geq 2$ be an integer, and let O be the $n \times n$ matrix whose entries are all equal to 0. Two distinct entries of the matrix are chosen uniformly at random, and those two entries are changed from 0 to 1. Call the resulting matrix A.

Determine the probability that $A^2 = O$, as a function of n.

B2. Let n be a positive integer. There are n lamps, each with a switch that changes the lamp from on to off, or from off to on, each time it is pressed. The lamps are initially all off.

You are going to press the switches in a series of rounds. In the first round, you are going to press exactly 1 switch; in the second round, you are going to press exactly 2 switches; and so on, so that in the kth round you are going to press exactly k switches. In each round you will press each switch at most once. Your goal is to finish a round with all of the lamps switched on.

Determine for which n you can achieve this goal.

B3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following two properties.

(i) The Riemann integral $\int_a^b f(t)\,\mathrm{d}t$ exists for all real numbers $a < b$.

(ii) For every real number x and every integer $n \geq 1$ we have

$$f(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\,\mathrm{d}t.$$

B4. *The following problem is open in the sense that the answer to part (b) is not currently known. A proof of part (a) will be awarded 5 points. Up to 7 additional points may be awarded for progress on part (b).*

Let $p(x)$ be a polynomial of degree d with coefficients belonging to the set of rational numbers $\mathbb{Q}$. Suppose that, for each $1 \leq k \leq d-1$, $p(x)$ and its kth derivative $p^{(k)}(x)$ have a common root in $\mathbb{Q}$; that is, there exists $r_k \in \mathbb{Q}$ such that $p(r_k) = p^{(k)}(r_k) = 0$.

(a) Prove that if d is prime then there exist constants $a, b, c \in \mathbb{Q}$ such that

$$p(x) = c(ax+b)^d.$$

(b) For which integers $d \geq 2$ does the conclusion of part (a) hold?

