SIMON MARAIS MATHEMATICS COMPETITION

2021
SOLUTIONS

Problem A1

Let a, b, c be real numbers such that $a \neq 0$. Consider the parabola with equation

$$y = ax^2 + bx + c,$$

and the lines defined by the six equations

$$\begin{array}{lll} y = ax + b, & y = bx + c, & y = cx + a, \\ y = bx + a, & y = cx + b, & y = ax + c. \end{array}$$

Suppose that the parabola intersects each of these lines in at most one point.

Determine the maximum and minimum possible values of $\dfrac{c}{a}$.

Solution

We will show that the maximum possible value of $\dfrac{c}{a}$ is 5, and the minimum possible value is 1.

First note that any linear scaling of the tuple (a, b, c) results in a quadratic which still satisfies the hypothesis of the problem, and also doesn't affect the value of $\frac{c}{a}$. Without loss of generality we may assume that $a = 1$, so the quadratic is

$$y = x^2 + bx + c.$$

First consider the intersection of $y = x^2 + bx + c$ and $y = x + c$. Solving $x^2 + bx + c = x + c$ gives us $x = 0$ and $x = (1 - b)$. These solutions must be equal and thus $b = 1$. Now the quadratic must be

$$y = x^2 + x + c.$$

Next consider the intersection of $y = x^2 + x + c$ and $y = cx + 1$. Solving $x^2 + x + c = cx + 1$ leads us to the quadratic $x^2 + (1 - c)x + (c - 1) = 0$. The discriminant of this quadratic must be nonpositive in order for it to have no more than one real solution, and thus

$$0 \geq (1 - c)^2 - 4(c - 1)$$

$$
\begin{aligned}
&= c^2 - 6c + 5 \\
&= (c-5)(c-1).
\end{aligned}
$$

Hence $1 \leq c \leq 5$, and therefore $1 \leq \frac{c}{a} \leq 5$.

Finally we also need to check that $y = x^2 + x + 1$ and $y = x^2 + x + 5$ both work.

- $y = x^2 + x + 1$ is tangent to $y = x + 1$ at $(0, 1)$.

Therefore $y = x^2 + x + 1$ works and so the minimal value of $\frac{c}{a} = 1$ is possible.

- $y = x^2 + x + 5$ is tangent to $y = x + 5$ at $(0, 5)$.
- $y = x^2 + x + 5$ is tangent to $y = 5x + 1$ at $(2, 11)$.
- $y = x^2 + x + 5$ does not intersect $y = x + 1$.

Therefore $y = x^2 + x + 5$ works and so the maximal value of $\frac{c}{a} = 5$ is possible.

Problem A2

Define the sequence of integers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, and

$$a_{n+1} = \big(n + 1 - \gcd(a_n, n)\big) \times a_n$$

for all integers $n \geq 1$.

Prove that $\dfrac{a_{n+1}}{a_n} = n$ if and only if n is prime or $n = 1$.

Here $\gcd(s, t)$ *denotes the greatest common divisor of* s *and* t.

Solution

Let $P(n)$ be the proposition that a_n is divisible by all primes less than n and is not divisible by any prime greater than or equal to n. We will prove $P(n)$ for all positive integers n by induction.

Clearly $P(1)$ holds. Suppose that $P(n)$ holds for some positive integer n.

- If n is prime or $n = 1$, then $\gcd(a_n, n) = 1$ and we have

  $$a_{n+1} = \big(n + 1 - \gcd(a_n, n)\big) \times a_n = n \times a_n.$$

  It follows that $P(n + 1)$ holds.

- If $n > 1$ and n is not prime, then $2 \leq \gcd(a_n, n) \leq n$ and we have that $1 \leq n + 1 - \gcd(a_n, n) < n$. Combine this with the induction hypothesis to obtain the fact that $a_{n+1} = \big(n + 1 - \gcd(a_n, n)\big) \times a_n$ is divisible by all primes less than n and is not divisible by any prime greater than or equal to n. It follows that $P(n + 1)$ holds.

Since $P(n)$ implies $P(n+1)$ in either case, we have proved that $P(n)$ holds for all positive integers n by induction.

Now if n is prime or $n = 1$, then $P(n)$ implies $\gcd(a_n, n) = 1$. Therefore, $\frac{a_{n+1}}{a_n} = n + 1 - \gcd(a_n, n) = n$. Conversely, if $\frac{a_{n+1}}{a_n} = n$, then $\gcd(a_n, n) = 1$. Combining this with $P(n)$ yields the fact that n is prime or $n = 1$.

Problem A3

Let $\mathcal{M}$ be the set of all 2021×2021 matrices with at most two entries in each row equal to 1 and all other entries equal to 0.

Determine the size of the set $\{\det A : A \in \mathcal{M}\}$.

Here $\det A$ *denotes the determinant of the matrix* A.

Solution

Let D be the set of possible determinants. We claim that

$$D = \{0, \pm 2^0, \pm 2^1, \pm 2^2, \pm 2^3, \ldots, \pm 2^{673}\},$$

so the number of possibilities is $2 \times 674 + 1 = 1349$.

To prove this we may argue as follows. First note that the determinant is clearly an integer and if $d \in D$, then by exchanging any two rows we have $-d \in D$. So take such a 2021×2021 matrix and let us consider the absolute value of its determinant.

If there is a row with zero 1s, then the determinant is 0. Otherwise, successively remove rows with one 1, along with the column containing that 1. By Laplace expansion, we obtain a matrix with the same determinant, up to sign. It is possible that we end up with a 1×1 matrix whose determinant is 0 or 1. Otherwise, we have a matrix that has two 1s in each row.

Now interpret each column of the resulting matrix as a vertex of a graph and each row as an edge between the vertices whose corresponding columns contain 1s. The graph cannot have loops, but can have multiple edges. If there is an isolated vertex in this graph, then the matrix has a column of 0s and hence the determinant is 0. Otherwise, remove all degree 1 vertices and their adjacent edges. This corresponds to applying Laplace expansion down a column with one 1 and produces a matrix with the same determinant, up to sign.

The resulting graph has n vertices and n edges for $n \leq 2021$, without any degree 0 or degree 1 vertices. Since the sum of the degrees is twice the number of edges, it must be the case that every vertex has degree 2. Therefore the graph is a disjoint union of cycles. This means that the rows and columns of the corresponding matrix can be permuted to give a block diagonal structure, with each block corresponding to a component of the graph. We may do this in such a way that each block is a matrix J_m for some integer $m \geq 2$, where the (i, i) and $(i, i+1)$ entries are 1 (with indices taken modulo m) and the remaining entries are 0.

The determinant of J_m is 0 for m even, since the sum of the even rows is equal to the sum of the odd rows. The determinant of J_m is 2 for m odd, by Laplace expansion down the first column. The two minors involved in this Laplace expansion are upper triangular matrices with 1s along the diagonal, and m being odd ensures that they contribute with the same sign.

It follows that the absolute value of the determinant is 0 if the graph contains any even cycles, and otherwise it is 2 to the power of the number of cycles in the graph. Odd cycles must have length at least 3, so since $2021 = 3 \times 673 + 2$ there can be at most 673 of them. For $0 \leq k \leq 673$ we may obtain a matrix with determinant 2^k by taking k blocks equal to J_3 and placing 1s in the remaining diagonal entries, and this completes the proof.

Problem A4

For each positive real number r, define $a_0(r) = 1$ and $a_{n+1}(r) = \lfloor ra_n(r) \rfloor$ for all integers $n \geq 0$.

(a) Prove that for each positive real number r, the limit

$$L(r) = \lim_{n\to\infty} \frac{a_n(r)}{r^n}$$

exists.

(b) Determine all possible values of $L(r)$ as r varies over the set of positive real numbers.

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Solution

We will solve part (a) in the process of proving that the answer to part (b) is the set $\mathcal{L} = \{0\} \cup (1/2, 1]$.

Define $b_n(r) = a_n(r)/r^n$. Then since $\lfloor x \rfloor$ satisfies $x - 1 < \lfloor x \rfloor \leq x$ we have $ra_n(r) - 1 < a_{n+1}(r) \leq ra_n(r)$, and hence

$$b_n(r) - \frac{1}{r^{n+1}} < b_{n+1}(r) \leq b_n(r). \tag{1}$$

It follows that the sequence $(b_n(r))_{n=0}^{\infty}$ is monotone decreasing. All terms are nonnegative so it is bounded below, and hence converges by the Monotone Convergence Theorem. This proves that the limit

$$L(r) = \lim_{n\to\infty} \frac{a_n(r)}{r^n}$$

exists, and moreover that $L(r) \leq b_n(r)$ for all n. This solves part (a), and further shows that $L(r) \leq b_0(r) = 1$.

In the case where r is an integer, $a_n(r) = r^n$ for all n, so $L(r) = 1$.

For $r \in (0, 1)$, we have $a_n(r) = 0$ for all n, so $L(r) = 0$. For $r \in (1, 2)$ we have $a_n(r) = 1$ for all n, so $L(r) = 0$.

Now consider $r > 2$, $r \notin \mathbb{Z}$. From the inequality (1) we get

$$b_{n+k}(r) > b_n(r) - \sum_{j=n+1}^{n+k} \frac{1}{r^j},$$

and hence

$$L(r) \geq b_n(r) - \sum_{j=n+1}^{\infty} \frac{1}{r^j} = b_n(r) - \frac{1}{r^n(r-1)}.$$

For $r > 3$ we use the case $n = 1$ together with the fact that $a_1(r) \geq r - 1$ to get

$$L(r) \geq b_1(r) - \frac{1}{r(r-1)} \geq \frac{r-1}{r} - \frac{1}{r(r-1)} = 1 - \frac{1}{r-1} > \frac{1}{2}.$$

For $2 < r < 3$ we have $b_1 = 2/r$, so

$$L(r) - \frac{1}{2} \geq \frac{2}{r} - \frac{1}{r(r-1)} - \frac{1}{2} = \frac{(3-r)(r-2)}{2r(r-1)} > 0.$$

This completes the proof that $L(r) \in \mathcal{L}$ for all r.

It remains to show that every $s \in \{0\} \cup (1/2, 1]$ is equal to $L(r)$ for some real $r > 0$. We have already seen that this is true for $s = 0, 1$, so it remains to show this for $s \in (1/2, 1)$. We will do this by proving the following theorem:

Theorem 1. *Given $s \in (1/2, 1)$, there exists $r \in (2, 3)$ such that $L(r) = s$.*

We give two proofs of this theorem. We first prove a lemma common to both.

Lemma 1. *For $r \in (2, 3)$ we have*

$$a_n(r) \leq \frac{3^n + 1}{2}.$$

Proof. The proof is by induction on n. For $n = 0$ we have $a_0 = 1 = \frac{3^0+1}{2}$, so the lemma is true in the base case. Suppose the lemma holds for some $n \geq 0$. Then

$$ra_n < 3a_n \leq \frac{3^{n+1} + 3}{2} = \frac{3^{n+1} + 1}{2} + 1.$$

Since the right hand side is an integer, on taking the floor of the left hand side we get

$$a_{n+1} = \lfloor ra_n \rfloor \leq \frac{3^{n+1} + 1}{2},$$

as required. □

Proof 1 of Theorem 1. We will use the following properties of the functions b_n.

1. Each function b_n is right continuous, meaning that

   $$\lim_{r \to r_0^+} b_n(r) = b_n(r_0).$$

   This follows by induction on n, using the fact that the floor function is right continuous.

2. Each function b_n is piecewise continuous, and decreasing on intervals of continuity. All discontinuities are jump discontinuities where the function jumps up.

   This follows from the fact that each function a_n is nondecreasing and piecewise constant, and $b_n(r) = a_n(r)/r^n$.

3. Given $r_0 \in [2,3)$, the function b_n takes every value between $b_n(r_0)$ and $\frac{3^n+1}{2\cdot 3^n}$ on $(r_0, 3)$.

   This follows from the previous point and Lemma 1 above, which gives

$$b_n(r) \le \frac{3^n+1}{2r^n}$$

   on $(2,3)$. This implies

$$\lim_{r\to 3^-} b_n(r) \le \frac{3^n+1}{2\cdot 3^n}.$$

Combining the first two points we have

$$\lim_{r\to r_0^-} b_n(r) \le b_n(r_0) = \lim_{r\to r_0^+} b_n(r). \tag{2}$$

Now fix $s \in (1/2, 1)$, and define sets X_n by

$$X_n = \left\{ r \in (2,3) : s \le b_n(r) \le s + \frac{1}{r^n - r^{n-1}} \right\}.$$

By point 3 above X_n is certainly nonempty for all n sufficiently large that $\frac{3^n+1}{2\cdot 3^n} < s$. Suppose that $r \in X_{n+1}$. Then from equation (1) we have

$$\begin{aligned} s \le b_{n+1}(r) \le b_n(r) < b_{n+1}(r) + \frac{1}{r^{n+1}} \\ &\le s + \frac{1}{r^{n+1} - r^n} + \frac{1}{r^{n+1}} \\ &= s + \frac{1}{r^n} \cdot \frac{2r-1}{r(r-1)} \\ &< s + \frac{1}{r^n - r^{n-1}} \qquad \text{(since } 2r-1 < r^2\text{)}. \end{aligned}$$

It follows that $X_{n+1} \subseteq X_n$ for all n. Combining this with the fact that X_n is nonempty for all sufficiently large n we conclude that X_n is nonempty for all n. Therefore $\{\overline{X_n} : n \in \mathbb{N}\}$ is a nested sequence of nonempty compact sets, hence has nonempty intersection.

Let $X = \bigcap_n \overline{X_n}$, and let $x = \sup X$. Then for all n either $x \in X_n$, or there is a monotone sequence in X_n converging to x. In either case we get $b_n(x) \ge s$ (using equation (2) in the second case), so $L(x) \ge s$. We claim that in fact $L(x) = s$.

If to the contrary we have $L(x) > s$ then we may define sets Y_n by

$$Y_n = X_n \cap (x,3) = \left\{ r \in (x,3) : s \le b_n(r) \le s + \frac{1}{r^n - r^{n-1}} \right\}.$$

Then by the same argument as above $\{\overline{Y_n} : n \in \mathbb{N}\}$ is a nested sequence of nonempty compact sets, so $Y = \bigcap_n \overline{Y_n}$ is nonempty. Moreover $Y_n \subseteq X_n$ for all n, so $Y \subseteq X$. Since x is a lower bound of Y and an upper bound of X we must have $Y = \{x\}$. It follows that x lies in the boundary of Y_n for each n.

Let $(r_{n,k})_{k\in\mathbb{N}}$ be a sequence in Y_n converging to x. Then $r_{n,k}$ converges to x from above, so by right continuity of b_n we have

$$s \le b_n(x) \le s + \frac{1}{r^n - r^{n-1}}$$

for all n. It follows that $L(x) = \lim_{n\to\infty} b_n(x) = s$, and we are done. □

Second proof of Theorem 1. We prove two further lemmas, and then show that together with Lemma 1 they imply the theorem.

Lemma 2. *Given $s \in (1/2, 1)$ there exists $\hat{r} \in (2,3)$ satisfying $L(\hat{r}) < s$.*

Proof. Since

$$\lim_{n\to\infty} \frac{3^n+1}{2\cdot 3^n} = \frac{1}{2} < s,$$

there exists an integer $N > 0$ such that

$$\frac{3^N+1}{2\cdot 3^N} < s.$$

Since $\frac{3^N+1}{2\cdot r^N}$ is continuous we may choose $\hat{r} \in (2,3)$ sufficiently close to 3 so that

$$\frac{3^N+1}{2\cdot \hat{r}^N} < s.$$

Then

$$L(\hat{r}) \le b_N(\hat{r}) = \frac{a_N(\hat{r})}{\hat{r}^N} \le \frac{3^N+1}{2\cdot \hat{r}^N} < s.$$

□

Lemma 3. *For any $r \in [2,3)$, the following holds:*

$$\limsup_{u\to r^-} L(u) \le L(r) \le \liminf_{u\to r^+} L(u).$$

Proof. For any $u, v \in \mathbb{R}_{>1}$ satisfying $u < v$ and any integer $n > 0$, we have the series of inequalities

$$L(v) \ge \frac{a_n(v)}{v^n} - \frac{1}{v^n(v-1)} \ge \frac{a_n(u)}{v^n} - \frac{1}{v^n(v-1)} \ge \frac{u^n L(u)}{v^n} - \frac{1}{v^n(v-1)}.$$

Hence,

$$L(v) \ge \lim_{n\to\infty} \limsup_{u\to v^-} \left[\frac{u^n L(u)}{v^n} - \frac{1}{v^n(v-1)}\right] = \limsup_{u\to v^-} L(u).$$

Moreover,

$$\liminf_{v\to u^+} L(v) \ge \lim_{n\to\infty} \liminf_{v\to u^+} \left[\frac{u^n L(u)}{v^n} - \frac{1}{v^n(v-1)}\right] = L(u).$$

This completes the proof of Lemma 3. □

To complete the proof of Theorem 1, fix $s \in (1/2, 1)$. It suffices to show that there is some $r \in (2, 3)$ satisfying $L(r) = s$. Since $2 \in \mathbb{Z}$ we have $L(2) = 1$, and by Lemma 2 there is some $\hat{r} \in (2, 3)$ satisfying $L(\hat{r}) < s$. Now let r be the infimum of the set

$$X = \{x \in (2, 3) : L(x) \leq s\}.$$

Then $\hat{r} \in X$, so $r \leq \hat{r}$. By the definition of r, either $L(r) \leq s$ or $\liminf_{u \to r^+} L(u) \leq s$. In either case $L(r) \leq s$, using Lemma 3 in the second case. In particular, this implies that $r \neq 2$, so $r \in (2, 3)$. For $u < r$, we have $L(u) > s$, so $\limsup_{u \to r^-} L(u) \geq s$. Hence, by Lemma 3, $L(r) \geq s$. This implies that $L(r) = s$, which completes the proof. $\square$

Problem B1

Let $n \geq 2$ be an integer, and let O be the $n \times n$ matrix whose entries are all equal to 0. Two distinct entries of the matrix are chosen uniformly at random, and those two entries are changed from 0 to 1. Call the resulting matrix A.

Determine the probability that $A^2 = O$, as a function of n.

Solution

Solution 1.

Suppose the (i, j)- and the (p, q)-entries are turned from 0 to 1 to form the matrix A. Note that $A^2 = O$ if and only if the dot product of any row of A with any column of A is equal to 0. This is equivalent to $\{i, p\} \cap \{j, q\} = \emptyset$.

Hence, if the two entries to be turned from 0 to 1 are picked one by one, the first one must be an entry not on the main diagonal. There are $n^2 - n$ ways to choose this out of n^2 possibilities. If the first one is the (i, j)-entry with $i \neq j$, then the second entry must, in addition to not being on the main diagonal, not be in the i-th column or the j-th row. Hence, out of the remaining $n^2 - 1$ possibilities, only $(n^2 - 1) - n - (2n - 3) = n^2 - 3n + 2$ will work; here, the 'minus n' is for the diagonal entries, while the 'minus $2n - 3$' is for off-diagonal entries in row j union those in column i. It follows that the answer is

$$\frac{n^2 - n}{n^2} \times \frac{n^2 - 3n + 2}{n^2 - 1} = \frac{(n-1)(n-2)}{n(n+1)}.$$

Solution 2.

Notice that

$$\frac{n^2 - n}{n^2} \times \frac{n^2 - 3n + 2}{n^2 - 1} = \frac{(n-1)^2(n^2 - 2n)}{n^2(n^2 - 1)} = \frac{\binom{(n-1)^2}{2}}{\binom{n^2}{2}}.$$

We give a bijective proof of the answer in this form.

Let $\mathcal{A}_n$ be the set of all $n \times n$ matrices that can be obtained from O by changing two distinct entries from 0 to 1, and let

$$\mathcal{B}_n = \{B \in \mathcal{A}_n : B^2 = O\}.$$

Clearly $|\mathcal{A}_n| = \binom{n^2}{2}$. We exhibit a bijection from $\mathcal{A}_{n-1}$ to $\mathcal{B}_n$, showing that $|\mathcal{B}_n| = \binom{(n-1)^2}{2}$.

Let $A \in \mathcal{A}_{n-1}$, and suppose that the nonzero entries of A are the (i, j)- and (p, q)-entries. We define $\phi(A) \in \mathcal{A}_n$ according to the series of mutually exclusive cases in Table 1. In all cases it can be seen that the resulting nonzero entries $(k, \ell), (r, s)$ of $\phi(A)$ satisfy $\{k, r\} \cap \{\ell, s\} = \emptyset$, so ϕ maps $\mathcal{A}_{n-1}$ into $\mathcal{B}_n$. We verify that it's one-to-one and onto.

Description of $A \in \mathcal{A}_{n-1}$	Strict conditions	Entries in A	Entries in $\phi(A)$
One diagonal entry	$i = j$ or $p = q$	$(i, i), (p, q)$	$(n, i), (p, q)$
One row/column shared	$i = q$ or $p = j$	$(i, j), (j, q)$	$(i, n), (j, q)$
Two diagonal entries	$i = j$ and $p = q$	$(i, i), (p, p)$	$(i, n), (p, n)$
Symmetric entries	$i = q$ and $p = j$	$(i, j), (j, i)$	$(n, j), (n, i)$
One diagonal and one row shared	$i = j$ and $p = i$	$(i, i), (i, q)$	$(i, n), (i, q)$
One diagonal and one column shared	$i = j$ and $q = j$	$(i, i), (p, i)$	$(n, i), (p, i)$
No diagonals, no row/column shared	$\{i, p\} \cap \{j, q\} = \emptyset$	$(i, j), (p, q)$	$(i, j), (p, q)$

Table 1: Cases used to define the bijection $\mathcal{A}_{n-1} \to \mathcal{B}_n$. In each row, there are no equalities between the indices i, j, p, q other than those that follow from the equations listed under "strict conditions". Each "or" should be understood as an exclusive or.

To see that ϕ is one-to-one, we check that the positions of the nonzero entries of $\phi(A)$ completely determine which row of the table A belongs to, and that the map defined in each row of the table is one-to-one.

To see that ϕ is onto we carry out a similar case by case check. Let $B \in \mathcal{B}_n$. If the nonzero entries of B belong to the upper left $(n-1) \times (n-1)$ submatrix B_{n-1} of B then $B = \phi(B_{n-1})$. So it remains to check that B belongs to the image of ϕ if one or both of the nonzero entries of B lie in the nth row or column. This is done by examining the final column of the table and verifying that all possible cases are covered.

Problem B2

Let n be a positive integer. There are n lamps, each with a switch that changes the lamp from on to off, or from off to on, each time it is pressed. The lamps are initially all off.

You are going to press the switches in a series of rounds. In the first round, you are going to press exactly 1 switch; in the second round, you are going to press exactly 2 switches; and so on, so that in the kth round you are going to press exactly k switches. In each round you will press each switch at most once. Your goal is to finish a round with all of the lamps switched on.

Determine for which n you can achieve this goal.

Solution

Note that the conditions imply that at most n rounds can take place. For $n = 2$ the goal cannot be achieved, because after the first round only one lamp will be on, and after the second round only the other lamp will be on. We show that the goal can be achieved for all other n.

We will present several solutions. Common to them all is that one or more small cases need to be handled separately, so we will treat all small cases that arise together first. We will regard the lamps as being numbered from 1 to n throughout.

Small cases.

The cases $1 \le n \le 6$, $n \ne 2$ may be handled as follows.

$n = 1$**:** The goal is achieved immediately in the first round.

$n = 3$**:** Press switch 1 in round one, and switches 2 and 3 in round two.

$n = 4$**:** Press switch 1 in round one; switches 2 and 3 in round two; switches 1, 2 and 3 in round three; and all four switches in round four.

$n = 5$**:** Press switch 1 in round one; switches 2 and 3 in round two; switches 1, 4 and 5 in round three; switches 2 to 5 in round four; and all five switches in round 5.

$n = 6$**:** Press switch 1 in round one; switches 2 and 3 in round two; and switches 4, 5 and 6 in round three. This achieves the goal for $n = 6$ in three rounds.

Solution 1.

Suppose that for some n the goal can be achieved in $k \le n$ rounds. For $n + 3$ lamps we may proceed as follows:

1. In the first k rounds, use the strategy for n to turn on the first n lamps.

2. In round $k+1$, turn off lamps 1 to k and turn on lamp $n+1$.

3. In round $k+2$, turn lamps 1 to k back on and turn on lamps $n+2$ and $n+3$.

This achieves the goal for $n+3$ in $k+2 < n+3$ rounds, proving that if the goal can be achieved for n it can be achieved for $n+3$. The small cases $n = 1$, 3 and 5 serve as the base for the induction to show that the goal can be achieved for all $n \neq 2$.

Solution 2.

Suppose that for some n the goal can be achieved in $k \leq n$ rounds. For $n+4$ lamps we may proceed as follows:

1. In the first k rounds, use the strategy for n to turn on the first n lamps.

2. In round $k+1$, turn off lamps 1 to k and turn on lamp $n+1$.

3. In round $k+2$, turn lamps 1 to k back on and turn on lamps $n+2$ and $n+3$.

4. In round $k+3$, turn lamps 1 to k and lamps $n+1$ to $n+3$ back off. Now lamps 1 to k and $n+1$ to $n+4$ are all off, and the remaining lamps are on.

5. In round $k+4$, turn lamps 1 to k and lamps $n+1$ to $n+4$ on, achieving the goal.

This achieves the goal for $n+4$ in $k+4 \leq n+4$ rounds, proving that if the goal can be achieved for n it can be achieved for $n+4$. The small cases $n = 1$, 3, 4 and 6 serve as the base for the induction to show that the goal can be achieved for all $n \neq 2$.

Comment.

Solutions 1 and 2 use special cases of the following more general result. Say that "(n, k) is possible" if the goal for n lamps can be achieved in k rounds. Then:

Lemma. *Suppose that (n, k) and (m, ℓ) are possible, and ℓ is even. Then $(n+m, k+\ell)$ is possible.*

Proof. In the first k rounds use the strategy for (n, k) to turn on the first n lamps. Choose k of these lamps, and in the remaining ℓ rounds flip these k lamps each round and use the remaining switch presses to turn on the last m lamps, following the strategy for (m, ℓ). □

Solution 1 uses the lemma with $(m, \ell) = (3, 2)$, and Solution 2 uses the lemma with $(m, \ell) = (4, 4)$. We may also use the lemma in its more general form to give a solution as follows.

Solution 3.

From our small cases above we know that $(1, 1)$, $(3, 2)$ and $(4, 4)$ are all possible. By the lemma applied to $(1, 1)$ and $(4, 4)$ we have that $(5, 5)$ is possible too.

Now note that for $n = 3$ and $n = 4$ the goal can be achieved in an even number of rounds. For any $n \geq 6$ we may write $n = s + t$, where s, t are positive integers such that $s, t \notin \{1, 2, 5\}$: we have $6 = 3 + 3$, $7 = 4 + 3$, $8 = 4 + 4$, and for $n \geq 9$ we may take for example $s = 3$, $t = n - 3$. Using the lemma it follows inductively that for any $n \geq 6$ the goal can be achieved for n lamps in an even number of rounds.

This proves that the goal can be achieved for all $n \neq 2$, and moreover that for $n \neq 1, 2, 5$ the goal can always be achieved in an even number of rounds. For completeness we note that for $n = 1$ and $n = 5$ the goal can be achieved in an odd number of rounds only. For $n = 1$ clearly only one round is possible. For $n = 5$ we can turn on at most three lamps in two rounds; and in four rounds we make a total of 10 switch presses, which means that at the end of the fourth round there will be an even number of lamps on, so at least one lamp must be off.

Solution 4.

If n is a triangular number, then clearly we are done. Otherwise, we have $n \geq 4$ and there exist unique positive integers m and k with $m \geq 2$ such that $n = (1 + 2 + \cdots + m) + k$ and $k \leq m$. So we can turn on all but k lamps in m rounds. Call those lamps turned on 'Group A' lamps and the rest 'Group B' lamps. Notice that there are at least $m + 1$ lamps in Group A, and at least $m + 3$ lamps in Group A if $m \geq 3$; and that $m + 2 \leq n$, and $m + 4 \leq n$ if $m \geq 3$.

If k is odd, say $k = 2h + 1$, then in round $m + 1$ we press the switches of any h lamps in Group B and any $m + 1 - h$ lamps in Group A, then in round $m + 2$ we press the switches of the remaining $h + 1$ lamps in Group B and the same $m + 1 - h$ lamps in Group A. This turns all the lamps on in $m + 2$ rounds and requires at most $m + 1$ lamps in Group A, so can always be done.

If k is even, we may find odd positive integers s and t such that $k = s + t$. Then we follow the same method as in the previous paragraph to first turn on s more lamps in rounds $m + 1$ and $m + 2$, and then turn on the remaining t lamps in rounds $m + 3$ and $m + 4$. This turns all the lamps on in $m + 4$ rounds and requires at most $m + 3$ lamps in Group A, so can always be done unless $m = 2$. The one outstanding case is $n = 5$, which is one of the small cases handled above.

Solution 5.

Notice that $\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$ and $\sum_{k=1}^{m+2} k = \frac{(m+2)(m+3)}{2}$ have opposite parities, because the difference $(m + 1) + (m + 2) = 2m + 3$ is odd. It follows that we can always choose $m = n$ or $m = n - 2$ such that $\sum_{k=1}^{m} k - n$ is even. Moreover, for $n \geq 4$ this choice of m satisfies $n - m \leq 2 \leq n - 2 \leq m$. We will show for $n \geq 4$ that the goal can be achieved in exactly m rounds.

To do this we will work backwards, starting with all of the lamps switched on and carrying out the rounds in reverse order, beginning with round m and working our way down to round 1. Our strategy will be to switch off as many lamps as possible in each round.

We first show by induction that after round k there are at most k lamps switched on. This is true in the base case $k = m$, because after round m there are $n - m$ lamps on, and $n - m \leq m$ by our observation above.

Now suppose that there are $\ell \leq k+1$ lamps on after round $k+1$, for some $1 \leq k \leq m-1$. If $\ell = k+1$ then in round k we will switch k of them off, finishing the round with exactly one lamp on. Otherwise, we have $0 \leq \ell \leq k$ and we will turn all ℓ of the lamps off and another $k - \ell$ lamps on. In either case we finish round k with at most k lamps turned on.

It follows that we finish round 1 with at most one lamp on. Now observe that there have been a total of $\sum_{k=1}^{m} k$ switch presses, we started with n lamps on, and each switch press changes the parity of the number of lamps that are on. Since $\sum_{k=1}^{m} k \equiv n \bmod 2$ by our choice of m, we must have also finished with an even number of lamps switched on. The only possibility is 0 lamps, so we have achieved the goal.

Problem B3

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following two properties.

(i) The Riemann integral $\displaystyle\int_a^b f(t)\,\mathrm{d}t$ exists for all real numbers $a < b$.

(ii) For every real number x and every integer $n \geq 1$ we have

$$f(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\,\mathrm{d}t.$$

Solution

Let $\mathcal{X}$ be the set of all functions satisfying the two properties. We claim that f belongs to $\mathcal{X}$ if and only if there exist real numbers p, q such that

$$f(x) = px + q$$

for all $x \in \mathbb{R}$; that is, $f \in \mathcal{X}$ if and only if f is affine.

It is easily verified that $\mathcal{X}$ contains all functions of this form, and that $\mathcal{X}$ is closed under taking real linear combinations (under usual pointwise addition and scalar multiplication). We present three proofs of the reverse inclusion.

Solution 1.

We prove the reverse inclusion through a series of lemmas.

Lemma 1. *Suppose that $f \in \mathcal{X}$. Then f is bounded on any closed interval: if $a, b \in \mathbb{R}$ with $a \leq b$, then there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.*

Proof. We will use the fact that if f is Riemann integrable on $[a, b]$ then so is $|f|$, and $\left|\int_a^b f(t)\,\mathrm{d}t\right| \leq \int_a^b |f(t)|\,\mathrm{d}t$. Given $x \in [a, b]$ we have

$$|f(x)| = \frac{1}{2}\left|\int_{x-1}^{x+1} f(t)\,\mathrm{d}t\right| \leq \frac{1}{2}\int_{x-1}^{x+1} |f(t)|\,\mathrm{d}t \leq \frac{1}{2}\int_{a-1}^{b+1} |f(t)|\,\mathrm{d}t,$$

which shows that f is bounded on $[a, b]$. □

Lemma 2. *Suppose that $f \in \mathcal{X}$. Then f is everywhere continuous.*

Proof. Fix an interval $[a, b]$ with $b - a > 3$, and choose M such that $|f| \leq M$ on $[a, b]$. Let $x, y \in [a + 1, b - 1]$ with $x < y < x + 2$. Then by taking $n = 1$, we have

$$|f(y) - f(x)| = \left|\frac{1}{2}\int_{y-1}^{y+1} f(t)\,\mathrm{d}t - \frac{1}{2}\int_{x-1}^{x+1} f(t)\,\mathrm{d}t\right|$$

$$
\begin{aligned}
&= \frac{1}{2}\left|-\int_{x-1}^{y-1} f(t)\,\mathrm{d}t + \int_{x+1}^{y+1} f(t)\,\mathrm{d}t\right| \\
&\le \frac{1}{2}\left|\int_{x-1}^{y-1} f(t)\,\mathrm{d}t\right| + \frac{1}{2}\left|\int_{x+1}^{y+1} f(t)\,\mathrm{d}t\right| \\
&\le \frac{1}{2}\int_{x-1}^{y-1} |f(t)|\,\mathrm{d}t + \frac{1}{2}\int_{x+1}^{y+1} |f(t)|\,\mathrm{d}t \\
&\le \frac{1}{2}(y-x)M + \frac{1}{2}(y-x)M \\
&= (y-x)M.
\end{aligned}
$$

It follows that f is continuous on $[a+1, b-1]$. Since $[a+1, b-1]$ can be chosen arbitrarily, f is continuous everywhere. □

Lemma 3. *Suppose that $f \in \mathcal{X}$ satisfies $f(a) = f(b) = 0$. Then f is identically 0 on $[a, b]$.*

Proof. Since f is continuous and $[a, b]$ is compact, we may define

$$M = \max_{x\in[a,b]} f(x) \ge 0.$$

We show that $M = 0$.

Suppose to the contrary that M is nonzero, and let $S = f^{-1}(M) \cap [a, b] \ne \emptyset$. Then S is closed, as the preimage of a closed set under a continuous map. We will show that S is also open in $[a, b]$, and hence $S = [a, b]$, because $[a, b]$ is connected. This contradicts the fact that $f(a) = f(b) = 0 \ne M$.

Given $x_0 \in S$ we have $x_0 \in (a, b)$, so we may choose a positive integer n such that $[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}] \subseteq [a, b]$. Let $g(x) = M - f(x)$ for all $x \in \mathbb{R}$. Then $g \in \mathcal{X}$, g is nonnegative on $[a, b]$, and $g(x_0) = 0$, so

$$0 = g(x_0) = \frac{n}{2}\int_{x_0-\frac{1}{n}}^{x_0+\frac{1}{n}} g(t)\ \mathrm{d}t.$$

Since g is nonnegative and continuous equality holds if and only if g is identically 0 on $[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$. We conclude that this interval is contained in S, so S is open. It follows from our discussion above that $M = 0$.

An identical argument shows that

$$\min_{x\in[a,b]} f(x) = 0$$

also, and the lemma is proved. □

Corollary 1. *Let $f \in \mathcal{X}$, and let $a, b \in \mathbb{R}$ with $a < b$. There exist real numbers p and q such that $f(x) = px + q$ for all $x \in [a, b]$.*

Proof. Choose $p, q \in \mathbb{R}$ such that

$$g(x) = px + q = \frac{f(b) - f(a)}{b - a}(x - a) + f(a);$$

that is, so that g is the line through $(a, f(a))$ and $(b, f(b))$. Let $h(x) = f(x) - g(x)$. Then $h \in \mathcal{X}$ and $h(a) = h(b) = 0$, so by our lemma above h is identically 0 on $[a, b]$. It follows that $f(x) = g(x) = px + q$ for all $x \in [a, b]$. □

Corollary 2. *Let $f \in \mathcal{X}$. Then there exist real numbers p and q such that $f(x) = px + q$ for all $x \in \mathbb{R}$, namely $p = f(1) - f(0)$ and $q = f(0)$.*

Proof. Given $x_0 \in \mathbb{R}$ choose $N > 1$ such that $x_0 \in (-N, N)$. From above there exist real numbers p, q such that $f(x) = px + q$ for all $x \in [-N, N]$. Then since $0, 1 \in [-N, N]$ we have

$$\begin{aligned} f(0) &= q, \\ f(1) &= p + q, \end{aligned}$$

and so $f(x) = (f(1) - f(0))x + f(0)$ for all $x \in [-N, N]$. In particular $f(x_0) = (f(1) - f(0))x_0 + f(0)$, completing the proof. □

Solution 2.

Let $f \in \mathcal{X}$. Then by Lemma 2 f is continuous, so by the Fundamental Theorem of Calculus the function

$$F(x) = \int_0^x f(t)\,\mathrm{d}t$$

is differentiable and $F'(x) = f(x)$. Since

$$\begin{aligned} f(x) &= \frac{1}{2}\int_{x-1}^{x+1} f(t)\,\mathrm{d}t \\ &= \frac{1}{2}\left(\int_0^{x+1} f(t)\,\mathrm{d}t - \int_0^{x-1} f(t)\,\mathrm{d}t\right) \\ &= \tfrac{1}{2}\left(F(x+1) - F(x-1)\right) \end{aligned}$$

it follows that f is differentiable, and moreover that

$$f'(x) = \tfrac{1}{2}(f(x+1) - f(x-1)).$$

Then since the right hand side is differentiable we further get

$$f''(x) = \tfrac{1}{4}(f(x+2) - 2f(x) + f(x-2)),$$

so f is twice differentiable and the second derivative is continuous. (In fact we can conclude that f is C^∞, but we will only need the fact that f is C^2.) Our goal is to show that f'' is identically zero.

Suppose that there exists $x_0 \in \mathbb{R}$ such that $f''(x_0) \neq 0$. By considering $-f \in \mathcal{X}$ if necessary we may assume without loss of generality that $f''(x_0) > 0$. By continuity of f'' there exists $n \in \mathbb{N}$ such that f'' is strictly positive on the interval $I = [x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$. Consider the function g defined by

$$g(x) = f(x) - f(x_0) - f'(x_0)(x - x_0).$$

Then $g \in \mathcal{X}$, $g(x_0) = g'(x_0) = 0$, and $g'' = f''$ is strictly positive on I. Applying standard calculus results it now follows that g is strictly positive on $I - \{x_0\}$. (For example, if $g(x_1) \leq 0$ for some $x_1 \in I$ with $x_1 \neq x_0$ then we may apply the Mean Value Theorem to g and then g' to get a point between x_0 and x_1 where g'' is nonpositive; or we may argue that $g'' > 0$ means g' is strictly increasing on I, so it changes sign from negative to positive at x_0, so g is strictly decreasing on $[x_0 - \frac{1}{n}, x_0]$ and strictly increasing on $[x_0, x_0 + \frac{1}{n}]$, and hence $g(x) > g(x_0) = 0$ for $x \in I - \{x_0\}$.) But then

$$g(x_0) = \frac{2}{n} \int_I g(t)\,\mathrm{d}t > 0,$$

a contradiction. So f'' must be identically zero on $\mathbb{R}$, as claimed.

It follows that there exist $p, q \in \mathbb{R}$ such that $f(x) = px + q$ for all $x \in \mathbb{R}$, as required.

Solution 3.

Let $n \geq 1$ be an integer and $x \in \mathbb{R}$. Then

$$\begin{aligned} 2f(x) &= n \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\ \mathrm{d}t \\ &= n \left(\int_{x-\frac{1}{n}}^{x} f(t)\ \mathrm{d}t + \int_{x}^{x+\frac{1}{n}} f(t)\ \mathrm{d}t \right) \\ &= f\left(x - \frac{1}{2n}\right) + f\left(x + \frac{1}{2n}\right). \end{aligned}$$

From this recurrence we get that there exist constants $a_n, b_n \in \mathbb{R}$ such that $f(x) = a_n x + b_n$ for all $x \in \frac{1}{2n}\mathbb{Z}$.

Since $|\frac{1}{2n}\mathbb{Z} \cap \frac{1}{2m}\mathbb{Z}| \geq 2$, we must have $a_n = a_m$ and $b_n = b_m$ for all m, n. Therefore there exist constants $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{Q}$.

Now for any $x \in \mathbb{R}$,

$$f(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\ \mathrm{d}t = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} (at + b)\ \mathrm{d}t = ax + b,$$

where in equating the two integrals, we used the fact that if two Riemann integrable functions agree on a dense set, then their integrals are the same (Proof: Both integrals are limits of Riemann sums where the points the function is evaluated at are restricted to the dense set). This completes the proof.

Problem B4

The following problem is open in the sense that the answer to part (b) is not currently known. A proof of part (a) will be awarded 5 points. Up to 7 additional points may be awarded for progress on part (b).

Let $p(x)$ be a polynomial of degree d with coefficients belonging to the set of rational numbers $\mathbb{Q}$. Suppose that, for each $1 \leq k \leq d-1$, $p(x)$ and its kth derivative $p^{(k)}(x)$ have a common root in $\mathbb{Q}$; that is, there exists $r_k \in \mathbb{Q}$ such that $p(r_k) = p^{(k)}(r_k) = 0$.

(a) Prove that if d is prime then there exist constants $a, b, c \in \mathbb{Q}$ such that

$$p(x) = c(ax+b)^d.$$

(b) For which integers $d \geq 2$ does the conclusion of part (a) hold?

Solution to part (a)

We begin by making several reductions. By replacing $p(x)$ with $p(x-r_i)$ for some $1 \leq i \leq d-1$ if necessary we may assume without loss of generality that 0 is a root of p. Hence we may write

$$p(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x.$$

Next, by replacing $p(x)$ with $p(cx)$ for a suitably chosen $c \in \mathbb{Q}$ if necessary, we may assume without loss of generality that the r_k are integers; and if they are not all zero, we may further assume that they have no common factor greater than 1. Finally, by replacing $p(x)$ with $cp(x)$ for a suitably chosen $c \in \mathbb{Q}$ if necessary we may additionally assume that the a_k are integers with no common factor greater than 1 as well. Our goal is to show that $a_k = 0$ for $1 \leq k \leq d-1$.

Now assume that d is prime. We first show that $a_k \equiv 0 \bmod d$ for $1 \leq k \leq d-1$. To do this, we will use the fact that $p^{(k)}(r_k) = 0$ if and only if the coefficient of x^k in $p(x+r_k)$ is equal to 0, because the coefficient of x^k in $q(x) = p(x+r_k)$ is

$$\frac{q^{(k)}(0)}{k!} = \frac{p^{(k)}(r_k)}{k!}.$$

Since

$$p(x+r_k) = \sum_{j=1}^{d} a_j (x+r_k)^j$$

this condition may be written as

$$\sum_{j=k}^{d} a_j \binom{j}{k} r_k^{j-k} = 0. \tag{3}$$

If there exists $1 \leq k \leq d-1$ such that $a_k \not\equiv 0 \bmod d$, then considering (3) mod d for the largest such k we have

$$a_k + a_d \binom{d}{k} r_k^{d-k} \equiv 0 \bmod d. \tag{4}$$

Since d is prime the binomial coefficient $\binom{d}{k}$ is 0 mod d, yielding $a_k \equiv 0 \bmod d$. Hence $a_k \equiv 0 \bmod d$ for $1 \leq k \leq d-1$, as claimed.

Now consider the equation $p(r_k) = 0 \bmod d$ for each k. By the previous paragraph we have

$$p(r_k) \equiv a_d r_k^d \equiv 0 \bmod d;$$

since d is prime this implies $d \mid a_d$ or $d \mid r_k$. But we can't have $d \mid a_d$, because this would violate our assumption that the coefficients a_j are relatively prime, so we must instead have $d \mid r_k$ for all k. But the r_k are also relatively prime, so this in turn implies $r_k = 0$ for all k. Then by our discussion above the coefficient of x^k in $p(x + r_k) = p(x)$ is equal to 0 for $1 \leq k \leq d-1$, so $p(x) = a_d x^d$, as required.

Comments on part (b)

This problem is the Casas-Alvero Conjecture in the case where the coefficient field is $\mathbb{Q}$. It is known to be true over fields of characteristic zero in certain special cases, including degrees that are a prime power or twice a prime power. We will address the case where d is a prime power, continuing to restrict ourselves to rational coefficients.

For $d = q^m$, q prime, the solution to part (a) above goes through essentially unchanged working mod q instead of mod d. The only point that requires comment is equation (4), which becomes

$$a_k + a_{q^m} \binom{q^m}{k} r_k^{q^m-k} \equiv 0 \bmod q.$$

Since q divides $\binom{q^m}{k}$ for $1 \leq k \leq q^m - 1$ we may conclude that $a_k \equiv 0 \bmod q$ for $1 \leq k \leq d-1$, as before. Continuing to work mod q the rest of the argument goes through unchanged and we get $p(x) = a_d x^d$, as required.

Once enough of the theory of non-archimedian valuations has been developed, the same proof can be used with any characteristic zero field in place of $\mathbb{Q}$.

