SIMON MARAIS
mathematics competition

2022 PAPER C: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions. Tablets may be used solely for writing worked solutions, with internet access switched off.
- For participants sitting the exam off-site, an electronic device such as a PC, laptop, phone or tablet may be used during the competition for accessing the papers, undergoing invigilation, writing and submitting solutions and (for pairs entrants) communicating with the other member of the pair.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "C3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and the paper (as in **S1234567C** (for singles) or **P3141593C** (for pairs)), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

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2022 PAPER C: PROBLEMS

C1. Let A and B be two fixed positive real numbers. The function f is defined by

$$f(x, y) = \min\left\{x, \ \frac{A}{y}, \ y + \frac{B}{x}\right\},$$

for all pairs (x, y) of positive real numbers.

Determine the largest possible value of $f(x, y)$.

C2. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\lceil \sin n \rceil}}$$

is convergent or divergent.

Here $\lceil x \rceil$ denotes the least integer greater than or equal to x.

C3. A grasshopper is sitting on the number line. Initially, it is sitting at the number 0. Each second it jumps one unit to the left or to the right, with equal probability. The directions of the jumps are chosen independently of each other.

Let p denote the probability that, after 2022 jumps, the grasshopper is sitting at a number divisible by 5. Determine whether $p < \frac{1}{5}$, $p = \frac{1}{5}$, or $p > \frac{1}{5}$.

C4. A machine is programmed to output a sequence of positive integers $a_1, a_2, a_3, \ldots$. It outputs the integers independently, one at a time, and at each step the probability that it outputs the integer k is equal to $\frac{1}{2^k}$. The machine stops when it outputs an integer that it has already output before.

Prove that the probability that the machine stops when it has output exactly n integers is

$$\frac{(n-1)!\,(2^n - n - 1)}{\prod_{r=1}^{n}(2^r - 1)}.$$