2022 SOLUTIONS

The solutions in this document are variously due to the proposers of the problems, members of the Problem Committee, members of the marking team, and participants in the competition.

Problem A1

Let $ABCD$ be a unit square, and let P be a point inside triangle ABC. Let E and F be the points on AB and AD, respectively, such that $AEPF$ is a rectangle. Let x and y denote the lengths of AE and AF, respectively.

Determine a two-variable polynomial $f(x,y)$ with the property that the area of $AEPF$ is greater than the area of each of the quadrilaterals $EBCP$ and $FPCD$ if and only if $f(x,y) > 0$.

Solution

The diagram is as shown in Figure 1. The area of $AEPF$ is xy, the area of $EBCP$ is

$$BE \cdot \frac{EP + BC}{2} = (1-x)\frac{y+1}{2} = \frac{(1-x)(1+y)}{2},$$

and by symmetry (or similar calculations), the area of $FPCD$ is $\frac{(1-y)(1+x)}{2}$.

It is given that P is inside triangle ABC, which implies $x > y$. The area of $FPCD$ minus the area of $EPCB$ is

$$\begin{aligned}\frac{(1-y)(1+x)}{2} - \frac{(1-x)(1+y)}{2} &= \frac{1+x-y-xy-(1+y-x-xy)}{2}\\ &= \frac{2x-2y}{2}\\ &= x-y > 0,\end{aligned}$$

so this means that the area of $AEPF$ is greater than the area of each of the other two quadrilaterals if and only if it is greater than the area of $FPCD$. (Note that if we allow P to lie on the boundary of ABC then we have $x \geq y$, and we reach the same conclusion here that the area of $AEPF$ is greater than the area of each of the other two quadrilaterals if and only if it is greater than the area of $FPCD$.)

Now

$$\text{area}(AEPF) > \text{area}(FPCD)$$

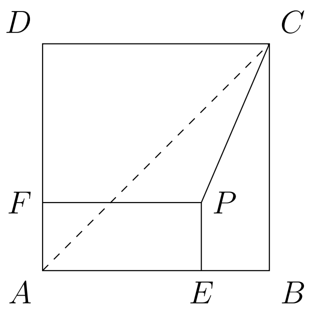


Figure 1: Diagram for Problem A1.

$$\begin{aligned}
\iff && xy &> \frac{(1-y)(1+x)}{2} \\
\iff && 2xy-(1-y)(1+x) &> 0 \\
\iff && 2xy-1-x+y+xy &> 0 \\
\iff && 3xy-x+y-1 &> 0,
\end{aligned}$$

so we can take $f(x, y) = 3xy - x + y - 1$.

Problem A2

Let n be a positive integer, and let S be a set with 2^n elements. Let $A_1, A_2, \ldots, A_n$ be randomly and independently chosen subsets of S, where each possible subset of S is chosen with equal probability. Let P_n be the probability that

$$A_1 \cup A_2 \cup \cdots \cup A_n = S \qquad \text{and} \qquad A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset.$$

Prove that $\lim_{n\to\infty} P_n = \dfrac{1}{e^2}$.

Solution

Note that for any $s \in S$ and $i \in \{1, 2, \ldots, n\}$, the probability that s belongs to A_i is always $\frac{1}{2}$. Now observe that

- $A_1 \cup A_2 \cup \cdots \cup A_n = S$ if and only if each $s \in S$ belongs to at least one A_i;
- $A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset$ if and only if each $s \in S$ does not belong to every A_i.

The probability that each $s \in S$ satisfies these two conditions is $1 - 2^{-n} - 2^{-n}$, where the first 2^{-n} is the probability that s does not belong to any A_i, while the second 2^{-n} is the probability that s belongs to every A_i. As S contains 2^n elements, the probability P_n is

$$P_n = (1 - 2^{1-n})^{2^n}.$$

So we need to determine

$$\lim_{n\to\infty} (1 - 2^{1-n})^{2^n}.$$

To do this we can make use of the known fact that

$$e^x = \lim_{k\to\infty} \left(1 + \frac{x}{k}\right)^k. \tag{1}$$

Writing our limit in the form

$$\lim_{n\to\infty} \left(1 + \frac{-2}{2^n}\right)^{2^n}$$

and observing that $2^n \to \infty$ as $n \to \infty$ we obtain $\lim_{n\to\infty} P_n = \dfrac{1}{e^2}$, as required.

To evaluate the limit without making use of the known limit (1) we may proceed as follows. Taking logs we have

$$\log P_n = 2^n \log(1 - 2^{1-n}) = \frac{\log(1 - 2^{1-n})}{2^{-n}}.$$

Applying L'Hôpital's rule, we have

$$\begin{aligned}\lim_{n\to\infty} \log P_n &= \lim_{n\to\infty} \frac{\log(1 - 2^{1-n})}{2^{-n}} \\ &= \lim_{n\to\infty} \frac{[\log(1 - 2^{1-n})]'}{(2^{-n})'}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \left(\frac{(\log 2)2^{1-n}}{(1-2^{1-n})} \cdot \frac{1}{-(\log 2)2^{-n}} \right) \\
&= \lim_{n\to\infty} \frac{-2}{1-2^{1-n}} \\
&= -2.
\end{aligned}
$$

Since $\lim_{n\to\infty} \log P_n$ exists and the exponential function is continuous we again find $\lim_{n\to\infty} P_n = \dfrac{1}{e^2}$, as required.

Problem A3

Let $0 < a < 1$ be a fixed real number. Show that there are at least two values of x in the interval $(0, 1)$ such that

$$\int_0^x \left(\sin\left(\frac{\pi \sin \frac{\pi t}{2}}{2} \right) + \frac{2}{\pi} \arcsin\left(\frac{2}{\pi} \arcsin t \right) - 2t \right) \mathrm{d}t = \frac{1}{2}\left(a - \frac{2}{\pi}\arcsin\left(\frac{2}{\pi}\arcsin a\right)\right)\left(\sin\left(\frac{\pi \sin\frac{\pi a}{2}}{2}\right) - a\right).$$

Solution

Observe that the functions

$$f(x) = \sin\left(\frac{\pi \sin \frac{\pi x}{2}}{2}\right)$$

and

$$g(x) = \frac{2}{\pi}\arcsin\left(\frac{2}{\pi}\arcsin x\right)$$

are inverse to each other on the interval $[0, 1]$ and hence their graphs are symmetric about the line $y = x$. Also, notice that $f(0) = g(0) = 0$, $f(1) = g(1) = 1$, both f and g are increasing, and that f is concave downward while g is concave upward.

Now let

$$F(x) = \int_0^x \big(f(t) + g(t) - 2t\big)\mathrm{d}t = \int_0^x \Big(\big(f(t) - t\big) - \big(t - g(t)\big)\Big)\mathrm{d}t$$

be the left-hand side of the equation, and let

$$A = \frac{(a - g(a))(f(a) - a)}{2}$$

be the right-hand side. Observe that since f and g are continuous on $[0, 1]$ so is F, by the Fundamental Theorem of Calculus.

Figure 2 shows the geometric meaning of $F(x)$ and A. It is clear that $F(0) = F(1) = 0$, that $A > 0$, and that $F(a) > A$. Therefore there is at least one solution of the equation $F(x) = A$ on the interval $(0, a)$, and at least one on the interval $(a, 1)$.

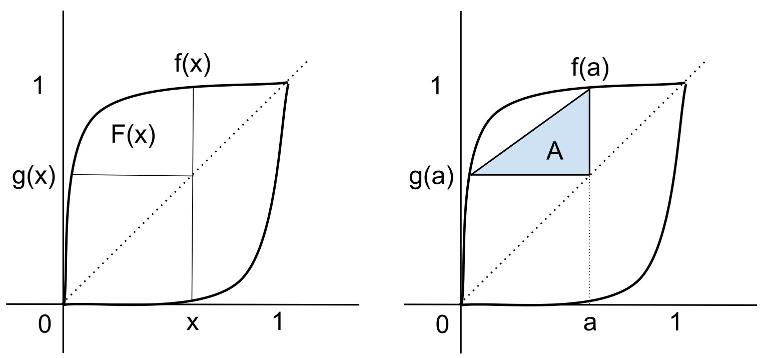


Figure 2: Graphs in Problem A3.

Problem A4

Let n be a positive integer, and let $q \geq 3$ be an odd integer such that every prime factor of q is larger than n. Prove that

$$\frac{1}{n!\,(q-1)^n}\prod_{i=1}^{n}(q^i-1)$$

is an integer that has no prime factor in common with $\dfrac{q-1}{2}$.

Solution

Let

$$X = \frac{1}{n!\,(q-1)^n}\prod_{i=1}^{n}(q^i-1) = \prod_{i=1}^{n}\frac{q^i-1}{i(q-1)}.$$

We give two proofs that X is an integer, and then show that it is relatively prime to $\dfrac{q-1}{2}$.

First proof that X is an integer

Since $\frac{q^i-1}{q-1} = \sum_{j=0}^{i-1} q^j$ is an integer for each i, to show that X is an integer we need to show that for each prime $p \leq n$, the product $\prod_{i=1}^{n}\frac{q^i-1}{q-1}$ has at least as many factors of p as $n!$ does. To do this it's sufficient to show that for each positive integer a there are at least as many factors in the product that are divisible by p^a as there are in $n! = \prod_{i=1}^{b} i$. This is because in each case we get a factor of p in the product for each pair of positive integers (a, j) such that p^a divides the factor for $i = j$.

Given a prime $p \leq n$ let p^b be the largest power of p dividing $q-1$, and for each positive integer a let d_a be the multiplicative order of q mod p^{a+b}. Then d_a is the smallest positive integer such that p^a divides $\frac{q^{d_a}-1}{q-1}$, and moreover p^a divides $\frac{q^d-1}{q-1}$ if and only if d_a divides d. We claim that $d_a \leq p^a$.

If $b = 0$ then d_a divides $\varphi(p^a)$, where φ is the Euler totient function. On the other hand, if $b > 0$ then q belongs to the subgroup of $U(\mathbb{Z}_{p^{a+b}})$ consisting of elements congruent to 1 modulo p^b, where $U(\mathbb{Z}_k)$ is the group of multiplicative units mod k. This subgroup has p^a elements, so d_a divides p^a. So in all cases $d_a \leq p^a$, as claimed.

The number of factors divisible by p^a in $\prod_{i=1}^{n}\frac{q^i-1}{q-1}$ is $\lfloor \frac{n}{d_a} \rfloor$, while the number of factors divisible by p^a in $n!$ is $\lfloor \frac{n}{p^a} \rfloor$. From $d_a \leq p^a$ we have $\lfloor \frac{n}{p^a} \rfloor \leq \lfloor \frac{n}{d_a} \rfloor$, so there are at least as many factors in the product $\prod_{i=1}^{n}\frac{q^i-1}{q-1}$ that are divisible by p^a as there are in $n!$. Since this is true for all a and all primes $p \leq n$, X is an integer.

Second proof that X is an integer

Since $q-1$ divides q^d-1 as a polynomial for all d, the statement only depends on the residue class of q modulo $n!$. Since every prime factor of q is greater than n, q is relatively prime to $n!$. By Dirichlet's theorem on primes in arithmetic progressions there exist infinitely many primes congruent to q mod $n!$, so we may assume without loss of generality that q is prime.

Let $G = GL_n(\mathbb{F}_q)$ and let N be the subgroup of monomial matrices (a matrix is a monomial matrix if it has exactly one nonzero entry in each row and column). Then

$$|G| = q^{\frac{n(n-1)}{2}} \prod_{d=1}^{n} q^d - 1 \qquad \text{and} \qquad |N| = n!(q-1)^n.$$

By Lagrange's theorem $|G|/|N|$ is an integer. Since q is relatively prime to $|N|$, we can further divide by the largest power of q in $|G|$ and deduce that

$$\frac{|G|}{q^{\frac{n(n-1)}{2}}|N|} = \frac{1}{n!(q-1)^n} \prod_{d=1}^{n} (q^d - 1)$$

is an integer.

Proof that X is relatively prime to $\dfrac{q-1}{2}$

Now let p be a prime dividing $\frac{q-1}{2}$ and let d be a positive integer. To conclude, it suffices to show that the p-adic valuation ν_p of the fraction

$$\frac{q^d - 1}{d(q-1)}$$

is zero. Write $q = 1 + 2m$. Then by the binomial theorem we have

$$\frac{q^d - 1}{d(q-1)} = \frac{1}{2dm} \sum_{i=1}^{d} \binom{d}{i} (2m)^i.$$

Let $a = \nu_p(d)$ and $b = \nu_p(m)$. Since $\nu_p(i!) = \lfloor \frac{i}{p} \rfloor + \lfloor \frac{i}{p^2} \rfloor + \lfloor \frac{i}{p^3} \rfloor + \cdots < \sum_{j=1}^{\infty} \frac{i}{p^j} = \frac{i}{p-1}$, we get

$$\nu_p \left(\binom{d}{i} (2m)^i \right) \geq \nu_p \left(\frac{d(2m)^i}{i!} \right) > a - \frac{i}{p-1} + i(b + \nu_p(2)).$$

If $p = 2$ then $\nu_p(2) - \frac{1}{p-1} = 0 = \frac{\nu_p(2)-1}{2}$, and if $p > 2$ then $\nu_p(2) - \frac{1}{p-1} = -\frac{1}{p-1} > -\frac{1}{2} = \frac{\nu_p(2)-1}{2}$. In either case $\nu_p(2) - \frac{1}{p-1} \geq \frac{\nu_p(2)-1}{2}$, so

$$\nu_p \left(\binom{d}{i} (2m)^i \right) > a + i \left(b + \frac{\nu_p(2) - 1}{2} \right).$$

For $i \geq 2$, we therefore get

$$\nu_p \left(\binom{d}{i} (2m)^i \right) > a + 2(b - \frac{1}{2}) = a + 2b + \nu_p(2) - 1 \geq a + b + \nu_p(2) = \nu_p(2dm),$$

as $b \geq 1$ from our assumption that p divides m. But $2dm = \binom{d}{1}(2m)^1$ is the term for $i = 1$, so this term has a strictly smaller p-adic valuation than every other term. It therefore determines the p-adic valuation of the sum, and we get

$$\nu_p \left(\frac{q^d - 1}{d(q-1)} \right) = \nu_p \left(\frac{2md}{2md} \right) = 0,$$

completing the proof.

Comment by the proposer (Peter McNamara)

Let G be a split reductive group over $\mathbb{F}_q$, which I conflate with its $\mathbb{F}_q$-points below in an abuse of notation. Let T be a maximal split torus and N its normaliser in G. Then

$$\frac{|G|}{|N|} = q^{|\Phi^+|} \prod_i \frac{q^{d_i} - 1}{d_i(q-1)}.$$

Here Φ^+ is the set of positive roots and the collection of integers $\{d_i\}$ are the exponents of the Weyl group. Then the same argument shows that this fraction is an integer, relatively prime to $\frac{q-1}{2}$.

If we remove the assumption that G is split, then I suspect the same conclusion is satisfied, but there is an additional argument needed as the formula for the quotient has additional factors.

I personally came across this result as I was interested in its consequence, namely that the mod $\frac{q-1}{2}$ cohomology groups of G and N are isomorphic.

Problem B1

Let $A > 1$ be a real number. Determine all pairs (m, n) of positive integers for which there exists a positive real number x such that $(1+x)^m = (1+Ax)^n$.

Solution

We show that the set of such pairs is $\{(m, n) : n < m < An\}$.

Solution 1.

Suppose first that there exists a positive real number x such that $(1+x)^m = (1+Ax)^n$. Then taking logs we have

$$m \log(1+x) = n \log(1+Ax),$$

which rearranges to

$$\frac{m}{n} = \frac{\log(1+Ax)}{\log(1+x)}.$$

Hence $n < m$, because $1 < 1+x < 1+Ax$ and so the left-hand side is greater than 1.

Next, to show that $m < An$, suppose to the contrary that $m \geq An$, and write $m = An+\alpha$ where $\alpha \geq 0$. Then $(1+x)^{An}(1+x)^{\alpha} = (1+Ax)^n$ implies $(1+Ax)^n \geq ((1+x)^A)^n$, so that $(1+Ax) \geq (1+x)^A$. However, this is not true if $x > 0$: for example, $y = 1+Ax$ is the tangent line to $f(x) = (1+x)^A$ at $x = 0$, and f lies above this line on $(0, \infty)$ because $f'' > 0$ there. Hence $m < An$.

To prove the converse assume that $n < m < An$. Then

$$\begin{aligned}
& (1+x)^m = (1+Ax)^n \\
\iff \quad & \sum_{k=0}^{m} \binom{m}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} A^k x^k \\
\iff \quad & \sum_{k=0}^{n} \left[\binom{n}{k} A^k - \binom{m}{k}\right] x^k - \sum_{k=n+1}^{m} \binom{m}{k} x^k = 0.
\end{aligned}$$

The polynomial on the left-hand side has constant term equal to zero, and after factoring out x, leaves a polynomial with constant term $An - m$ and leading term $-x^{m-1}$. But this polynomial has a positive real root x, as $An - m$ (the y-intercept) is positive, and the leading coefficient is negative. This completes the proof that the set of all desired pairs (m, n) is given by $\{(m, n) : n < m < An\}$, as claimed.

Solution 2.

We present a second method of proving the converse.

Define

$$f(x) = \frac{\log(1+Ax)}{\log(1+x)}.$$

Then f is continuous on $(0, \infty)$. We will show that f takes every value in $(1, A)$, by showing that

$$\lim_{x \to 0} f(x) = A, \qquad \lim_{x \to \infty} f(x) = 1.$$

First consider

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log(1+Ax)}{\log(1+x)}.$$

This is an indeteminate form of type $\frac{0}{0}$, so by L'Hôpital's Rule we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\frac{A}{1+Ax}}{\frac{1}{1+x}} = \lim_{x\to 0} \frac{A(1+x)}{1+Ax} = A.$$

Next consider $\lim_{x\to\infty} f(x)$. This time we have an indeterminate form of type $\frac{\infty}{\infty}$, so applying L'Hôpital's Rule once more we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{A(1+x)}{1+Ax} = 1.$$

By the Intermediate Value Theorem f takes every value in $(1, A)$ on $(0, \infty)$. In particular it takes every rational value in this interval, which is every rational number of the form $\frac{m}{n}$ for positive integers m, n with $n < m < An$. This completes the proof of the converse.

Problem B2

Let a, b, and c be real numbers, and let P be the polynomial

$$P(x) = x^6 + ax^4 + bx^2 + c.$$

Suppose that there is a unique circle Γ in the complex plane such that all of the roots of P lie on Γ. Prove that $b^3 = a^3c$.

Solution

Note that since $P(x)$ has real coefficients, the non-real roots occur in complex conjugate pairs. Moreover, since $P(x)$ is even, if w is a root then so is $-w$.

Now if the roots all lie on a line in $\mathbb{C}$, then the fact that they all lie on a circle implies that there are at most two distinct roots, which contradicts the uniqueness of Γ. Hence not all of the roots lie in $\mathbb{R}$ or in $i\mathbb{R}$.

We now split into two cases. In the first case, $P(x)$ has a root $w \in \mathbb{C}\backslash(\mathbb{R} \cup i\mathbb{R})$; in this case, $\overline{w}$, $-w$, and $-\overline{w}$ are also roots of $P(x)$, and these four are distinct. The unique circle Γ on which the six roots lie is thus the circle centred at 0 with radius $r = |w|$. Note that we cannot in addition to these four roots have another root $w' \in \mathbb{C}\backslash(\mathbb{R} \cup i\mathbb{R})$, since then $P(x)$ will have at least eight roots. Therefore the remaining two roots are either in $\mathbb{R}$, and thus $-r$ and r, or in $i\mathbb{R}$, and thus ir and $-ir$. Let $\varepsilon = +1$ if $\pm r$ are roots of P, and -1 if $\pm ir$ are roots of P. Then since P is monic we have

$$\begin{aligned}
P(x) &= (x - w)(x + w)(x - \overline{w})(x + \overline{w})(x^2 - \varepsilon r^2) \\
&= (x^2 - w^2)(x^2 - \overline{w}^2)(x^2 - \varepsilon|w|^2) \\
&= (x^4 - (w^2 + \overline{w}^2)x^2 + |w|^4)(x^2 - \varepsilon|w|^2) \\
&= x^6 - (w^2 + \overline{w}^2 + \varepsilon|w|^2)x^4 + \varepsilon|w|^2(w^2 + \overline{w}^2 + \varepsilon|w|^2)x^2 - \varepsilon|w|^6 \\
&= x^6 - \zeta x^4 + \varepsilon|w|^2\zeta x^2 - \varepsilon|w|^6,
\end{aligned}$$

for $\zeta = w^2 + \overline{w}^2 + \varepsilon|w|^2$. Then

$$a^3c = (-\zeta)^3(-\varepsilon|w|^6) = \varepsilon\zeta^3|w|^6 = (\varepsilon\zeta|w|^2)^3 = b^3,$$

and the result holds in this case.

In the second case, all the roots lie in $\mathbb{R}$ or in $i\mathbb{R}$, and not all lie on one of them. So we have at least two real roots r and $-r$ and at least two imaginary roots ir' and $-ir'$, where $r, r' > 0$. These four form a rhombus, which is cyclic if and only if $r = r'$; so we have roots r, $-r$, ir, and $-ir$. Since the remaining two roots also lie on the circle with radius r, they are either r and $-r$ or ir and $-ir$. Again letting $\varepsilon = +1$ if $\pm r$ are roots of P and -1 if $\pm ir$ are roots of P, we have

$$\begin{aligned}
P(x) &= (x^2 - r^2)(x^2 + r^2)(x^2 - \varepsilon r^2) \\
&= (x^4 - r^4)(x^2 - \varepsilon r^2) \\
&= x^6 - \varepsilon r^2x^4 - r^4x^2 + \varepsilon r^6.
\end{aligned}$$

Then

$$a^3c = (-\varepsilon r^2)^3(\varepsilon r^6) = -\varepsilon^4r^{12} = -r^{12} = (-r^4)^3 = b^3,$$

and the result holds in this case also.

Problem B3

Ari and Sam are playing a game in which they take turns breaking a block of chocolate in two and eating one of the pieces. At each stage of the game the block of chocolate is a rectangle with integer side lengths. On each player's turn, they break the block of chocolate into two such rectangles along a horizontal or vertical line, and eat the piece with smaller area. (If the two pieces have the same area they may eat either one.) The game ends when the block of chocolate is a 1×1 rectangle, and the winner is the last player to take their turn breaking the chocolate in two.

At the start of the game the block of chocolate is a 58×2022 rectangle. If Ari goes first, which player has a winning strategy?

Solution

We show that Ari has a winning strategy.

In what follows we will write (a, b) for the position in the game where the block of chocolate is an $a \times b$ rectangle. We will say that a position (a, b) is a *first-player-win* if the first player to move in that position has a winning strategy, and a *second-player-win* if the second player to move in that position has a winning strategy. These positions are also known as $\mathcal{N}$ (for *next player*) and $\mathcal{P}$ (for *previous player*) positions, respectively.

Solution 1

A win-loss analysis table can be built up leading to the conjecture that (a, b) is a second-player-win if and only if $\frac{b+1}{a+1}$ is an integer power of 2, that is, $\frac{b+1}{a+1} = 2^z$ for some $z \in \mathbb{Z}$. The proof is by induction on $a + b$.

The base case is $a + b = 2$ so that $(a, b) = (1, 1)$. This is a second-player-win by definition and we have $\frac{a+1}{b+1} = 2^0$.

Suppose that the conjecture is true for all (a, b) satisfying $a + b < n$, and let (a, b) be such that $a + b = n$.

Case 1. Suppose that $\frac{a+1}{b+1} \neq 2^z$ for any integer z.

Without loss of generality $a > b$. There exists an integer $k \geq 0$ such that

$$2^k(b + 1) < a + 1 < 2^{k+1}(b + 1) \quad \Leftrightarrow \quad 2^k(b + 1) \leq a \leq 2^{k+1}(b + 1) - 2.$$

The right part of the inequality implies $2^k(b + 1) - 1 \geq \frac{a}{2}$. Thus we may reduce (a, b) to $(2^k(b+1) - 1, b)$. By the inductive assumption this position is a second-player win. Hence (a, b) is a first-player-win.

Case 2. Suppose that $\frac{a+1}{b+1} = 2^z$ for some integer z.

Without loss of generality we may assume that the bar is broken into a $b \times r$ piece and a $b \times s$ piece where $r + s = a$ and $0 < r \leq s < a$. Note that $s \geq \frac{a}{2}$.

By the inductive assumption (s, b) is a second-player-win if and only if $\frac{s+1}{b+1} = 2^y$ for some integer y. Note that $s < a$ implies $2^y = \frac{s+1}{b+1} < \frac{a+1}{b+1} = 2^z$, and so $y \leq z - 1$. Dividing the two equations yields $\frac{s+1}{a+1} \leq 2^{y-z} \leq \frac{1}{2}$. Hence $2s \leq a - 1$, which contradicts $s \geq \frac{a}{2}$.

Therefore all possible moves from (a, b) are to first-player-wins. It follows that (a, b) is a second-player-win.

Since $\frac{2023}{59}$ is not of the form 2^z for any integer z, we conclude that $(58, 2022)$ is a first-player-win. Therefore Ari has a winning strategy.

Solution 2

Observe that the game is equivalent to normal play nim played with two heaps of beans, subject to the restriction that you may take at most half of the beans from a heap on your move. By the Sprague-Grundy Theorem each of the two heaps is equivalent to a single nim-heap of some size in nim played without this restriction. The size of this heap is the *Grundy value* or *nim-value* of the original heap. If we know these values then the game reduces to ordinary two-heap nim, where the second-player-wins are exactly the positions where the two heaps have the same size. Thus, we can solve the problem by finding the nim-values for the game played with a single nim heap, where you may take at most half of the beans from the heap on your move.

For convenience we will add a single bean to our heap, and change the rule so that on your move you must take strictly less than half of the beans. This modified game is in fact equivalent to our original one-heap game: in the original game you can move from $2m$ to any integer in the range m to $2m - 1$, while in the modified game you can move from $2m + 1$ to any integer in the range $m + 1$ to $2m$; and in the original game you can move from $2m + 1$ to any integer in the range $m + 1$ to $2m$, while in the modified game you can move from $2m + 2$ to any integer in the range $m + 2$ to $2m + 1$. We will find the nim-values for the modified game.

For each positive integer n let $f(n)$ be the nim-value of a heap of n beans, in the game where you must take strictly less than half of the beans in the heap on your move. The first two values are $f(1) = f(2) = 0$ (no move is possible), and by the Sprague-Grundy theory $f(n)$ is the mex (the *minimum excludant*) of the nim-values of the heaps it is possible to move to from n:

$$\begin{aligned} f(n) &= \text{mex}\{f(m) : m \text{ is a legal move from } n\} \\ &= \text{mex}\{f(m) : n/2 < m < n\}. \end{aligned}$$

That is, $f(n)$ is the least nonnegative integer that does *not* belong to the set $\{f(m) : n/2 < m < n\}$. Using this we may find for example that the first 10 values of $f(n)$ are given by the sequence

$$0, 0, 1, 0, 2, 1, 3, 0, 4, 2.$$

For example, $f(10) = 2$ because from a heap of 10 beans you can move to a heap of size $6, 7, 8, 9$; these have nim-values $1, 3, 0, 4$, respectively, so the mex is 2.

Each positive integer n may be written uniquely in the form $n = 2^s(2t+1)$, for nonnegative integers s and t. Using this representation define $g : \mathbb{N} \to \mathbb{N} \cup \{0\}$ by $g(n) = g(2^s(2t+1)) = t$. Although we won't use it in what follows, this function can be calculated by writing n in binary, and deleting the right-most 1 and all 0s to the right of it. We claim that $f(n) = g(n)$.

The proof is by induction on n. For the base case we have $g(1) = 0 = f(1)$ and $g(2) =$

$0 = f(2)$. So suppose that $f(k) = g(k)$ for all $1 \le k < n$. If $n = 2m+1$ is odd, then

$$\begin{aligned} f(n) = f(2m+1) &= \operatorname{mex}\{f(m+1), f(m+2), \ldots, f(2m)\} \\ &= \operatorname{mex}\{g(m+1), g(m+2), \ldots, g(2m)\}. \end{aligned}$$

We claim that

$$\{g(m+1), g(m+2), \ldots, g(2m)\} = \{0, 1, \ldots, m-1\}, \tag{2}$$

so that the mex is $m = g(2m+1)$. Indeed, suppose that $0 \le \ell < m$. Then $2\ell + 1 < 2m$, and if $2^s(2\ell+1) < m+1$ then $2^{s+1}(2\ell+1) < 2m+2$, which implies $2^{s+1}(2\ell+1) \le 2m$ since the left-hand side is even. It follows that the sequence $(2^s(2\ell+1)_{s\ge 0}$ must contain an integer belonging to the interval $[m+1, 2m]$. This shows that ℓ belongs to the set $\{g(m+1), g(m+2), \ldots, g(2m)\}$ for all $0 \le \ell < m$. Since the set has at most m elements, and there are m integers ℓ such that $0 \le \ell < m$, the equality (2) must hold. Then

$$f(2m+1) = \operatorname{mex}\{0, 1, \ldots, m-1\} = m = g(2m+1),$$

as required.

Now if $n = 2m+2$ is even then

$$\begin{aligned} f(n) = f(2m+2) &= \operatorname{mex}\{f(m+2), f(m+3), \ldots, f(2m), f(2m+1)\} \\ &= \operatorname{mex}\{g(m+2), g(m+3), \ldots, g(2m), g(2m+1)\}. \end{aligned}$$

We've shown above that $g(m+1), \ldots, g(2m)$ are the integers $0, \ldots, m-1$ in some order, and that $g(2m+1) = m$. It follows that $\{g(m+2), g(m+3), \ldots, g(2m), g(2m+1)\}$ is the set $\{0, 1, \ldots, m\}$ with exactly one element deleted, namely $g(m+1)$, and so the mex of this set is $g(m+1)$. But $g(m+1) = g(2m+2)$, so this completes the proof of the inductive step.

Returning now to Ari, Sam and the bar of chocolate, the position (a, b) is a second-player win if and only if $g(a+1) = g(b+1)$. This holds if and only if $\frac{a+1}{b+1}$ is a power of 2, as we found in the first solution.

Comments

1. The analysis in the second solution is longer than that in the first, but it tells us how to play the game with more heaps of beans as well. For example, it tells us how to play the same game with a block of chocolate that forms a rectangular prism with integer side lengths $a \times b \times c$. Such a block of chocolate is a win for the second player if and only if the nim sum $g(a+1) \oplus g(b+1) \oplus g(c+1)$ is 0, where nim addition is done by writing the numbers in base two and adding without carrying.

   For example, with $a = 58$ and $b = 2022$ we have

   $$g(a+1) \oplus g(b+1) = 59 \oplus 2023 = 2012,$$

   and the smallest value of c such that $g(c+1) = 2012$ is 4024. Consequently, for a $58 \times 2022 \times c$ block of chocolate with Ari moving first, the smallest value of c such that Sam has a winning strategy is 4024.

2. The function g above satisfies the recurrence relation $g(2k+1) = k$, $g(2k) = g(k)$. This gives the sequence $(g(n))_{n\geq 1}$ a self-similar fractal property, whereby the values at the odd integers are the nonnegative integers in order, and the values at the even integers (shown bolded) are a copy of the original sequence:

$$0, \mathbf{0}, 1, \mathbf{0}, 2, \mathbf{1}, 3, \mathbf{0}, 4, \mathbf{2}, 5, \mathbf{1}, 6, \mathbf{3}, 7, \mathbf{0}, 8, \ldots$$

This is an example of a *fractal sequence*, as defined by Kimberling [1, 2].

Levine [3] studies the game *Maximum Nim*, where there is a function ϕ such that a player may take at most $\phi(n)$ beans from a heap of size n. The nim-variant used in the second solution corresponds to $\phi(n) = \lfloor \frac{n-1}{2} \rfloor$. Levine shows that the nim-value sequences of Maximum Nim with a weakly increasing rule ϕ are precisely Kimberling's fractal sequences.

The sequence $(g(n+1))_{n\geq 0}$ appears in the Online Encyclopedia of Integer Sequences as sequence A025480.

Problem B4

Let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be two sequences of positive integers, satisfying $a_0 b_0 \geq 600$ and

$$a_{n+1} = a_n + 2 \cdot \lfloor b_n/20 \rfloor,$$
$$b_{n+1} = b_n + 3 \cdot \lfloor a_n/30 \rfloor,$$

for all $n \geq 0$.

(a) Prove that there exists a nonnegative integer N such that

$$-13 \leq a_n - b_n \leq 23,$$

for all $n \geq N$.

(b) Must there exist a nonnegative integer n such that $a_n = b_n$?

Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Solution to part (a)

Define $f : \mathbb{N}^2 \to \mathbb{N}^2$ by

$$f(a, b) = (a + 2 \cdot \lfloor b/20 \rfloor, b + 3 \cdot \lfloor a/30 \rfloor),$$

so that

$$(a_{n+1}, b_{n+1}) = f(a_n, b_n).$$

For $d \in \mathbb{Z}$ let

$$L_d = \{(a, b) \in \mathbb{N} : a - b \leq d\},$$
$$R_d = \{(a, b) \in \mathbb{N} : a - b \geq d\},$$

and note that if $d < d'$ then $L_d \subseteq L_{d'}$ while $R_{d'} \subseteq R_d$. Our goal is to show that eventually (a_n, b_n) is trapped in the region $R_{-13} \cap L_{23}$. We will begin by proving the following lemma, which is sufficient to show that (a_n, b_n) is eventually trapped in the region $R_{-19} \cap L_{29}$.

Lemma. *Let d be an integer.*

1. *If $d \geq 20$ then $f(L_d) \subseteq L_d$; and if $d \geq 30$ then $f(L_d) \subseteq L_{d-1}$.*
2. *If $d \leq -10$ then $f(R_d) \subseteq R_d$; and if $d \leq -20$ then $f(R_d) \subseteq R_{d+1}$.*

Corollary. *The set $R_m \cap L_M$ is an invariant set of f for any $m \leq -10$ and $M \geq 20$. Moreover it is an attracting invariant set of f if $m \leq -19$ and $M \geq 29$, in the sense that the orbit of any point under f must eventually enter $R_{-19} \cap L_{29}$ (and is then trapped there).*

Let

$$d_n = a_n - b_n$$

for all $n \geq 0$. Then from the defining equation of the sequence we have

$$\begin{aligned} d_{n+1} &= a_{n+1} - b_{n+1} \\ &= (a_n + 2 \cdot \lfloor b_n/20 \rfloor) - (b_n + 3 \cdot \lfloor a_n/30 \rfloor) \\ &= (a_n - b_n) + (2 \cdot \lfloor b_n/20 \rfloor - 3 \cdot \lfloor a_n/30 \rfloor) \\ &= d_n + (2 \cdot \lfloor b_n/20 \rfloor - 3 \cdot \lfloor a_n/30 \rfloor). \end{aligned}$$

Now using $x - 1 < \lfloor x \rfloor \leq x$ we obtain

$$\begin{aligned} a_n/10 - 3 < 3\lfloor a_n/30 \rfloor \leq a_n/10, \\ b_n/10 - 2 < 2\lfloor b_n/20 \rfloor \leq b_n/10, \end{aligned}$$

so

$$b_n/10 - a_n/10 - 2 < 2 \cdot \lfloor b_n/20 \rfloor - 3 \cdot \lfloor a_n/30 \rfloor < b_n/10 - a_n/10 + 3,$$

which says

$$-d_n/10 - 2 < 2 \cdot \lfloor b_n/20 \rfloor - 3 \cdot \lfloor a_n/30 \rfloor < -d_n/10 + 3.$$

Hence

$$\tfrac{9}{10}d_n - 2 < d_{n+1} < \tfrac{9}{10}d_n + 3.$$

Proof of lemma.

1. Suppose that $d_{n+1} > d_n$. Then we must have $d_{n+1} \geq d_n + 1$, so

   $$d_n + 1 < \tfrac{9}{10}d_n + 3,$$

   which rearranges to $d_n < 20$. However, if $d_n < 20$ then $d_{n+1} < 18 + 3 = 21$, so $d_{n+1} \leq 20$. It follows that $f(L_d) \subseteq L_d$ for $d \geq 20$.

   Now suppose that $d_n \geq 30$. Then $3 \leq d_n/10$, so $d_{n+1} < 9d_n/10 + 3 \leq d_n$, which implies $d_{n+1} \leq d_n - 1$ since d_m is an integer for all m. Combining this with $f(L_{29}) \subseteq L_{29}$ proved above shows that $f(L_d) \subseteq L_{d-1}$ for $d \geq 30$.

2. The proof is similar. Suppose that $d_{n+1} < d_n$. Then we must have $d_{n+1} \leq d_n - 1$, so

   $$\tfrac{9}{10}d_n - 2 < d_n - 1,$$

   which rearranges to $d_n > -10$. However, if $d_n > -10$ then $d_{n+1} > -9 - 2 = -11$, so $d_{n+1} \geq -10$. It follows that $f(R_d) \subseteq R_d$ for $d \leq -10$.

   Now suppose that $d_n \leq -20$. Then $d_n/10 \leq -2$, so $d_n = 9d_n/10 + d_n/10 \leq 9d_n/10 - 2 < d_{n+1}$. It follows that $d_{n+1} \geq d_n + 1$. Combining this with $f(R_{-19}) \subseteq R_{-19}$ proved above shows that $f(R_d) \subseteq R_{d+1}$ for $d \leq -20$.

□

To refine the above we take a second approach. Use the division algorithm to write

$$a_n = 30s_n + u_n,$$
$$b_n = 20t_n + v_n,$$

for unique integers s_n, t_n, u_n, v_n such that $0 \le u_n \le 29$ and $0 \le v_n \le 19$. Then

$$3 \cdot \lfloor a_n/30 \rfloor = 3s_n, \qquad 2 \cdot \lfloor b_n/20 \rfloor = 2t_n.$$

The lemma implies that if (d_n) is not eventually trapped in $[-10, 20]$ then it must be eventually constant, with constant value in $[-19, -11] \cup [21, 29]$. So we examine when we can have $\Delta_n = d_{n+1} - d_n = 0$. From above we have

$$\begin{aligned}\Delta_n &= d_{n+1} - d_n \\ &= 2 \cdot \lfloor b_n/20 \rfloor - 3 \cdot \lfloor a_n/30 \rfloor \\ &= 2t_n - 3s_n,\end{aligned}$$

so if $\Delta_n = 0$ then $3s_n = 2t_n = 6q_n$ for some positive integer q_n. Hence

$$a_n = 60q_n + u_n,$$
$$b_n = 60q_n + v_n,$$

where $0 \le u_n \le 29$ and $0 \le v_n \le 19$. Then

$$a_{n+1} = 66q_n + u_n,$$
$$b_{n+1} = 66q_n + v_n.$$

Write $q_n = 10^k m$ for nonnegative integers k and m such that $10 \nmid m$. Then

$$a_{n+1} = 60q_{n+1} + u_n,$$
$$b_{n+1} = 60q_{n+1} + v_n,$$

where $q_{n+1} = 10^{k-1} \times 11m$. After k steps we have

$$a_{n+k} = 60 \times 11^k m + u_n,$$
$$b_{n+k} = 60 \times 11^k m + v_n,$$

and then

$$a_{n+k+1} = 60 \times 11^k m + 6 \times 11^k m + u_n,$$
$$b_{n+k+1} = 60 \times 11^k m + 6 \times 11^k m + v_n.$$

Hence $a_{n+k+1} \equiv a_{n+k} \bmod 6$ but $a_{n+k+1} \not\equiv a_{n+k} \bmod 60$; and similarly $b_{n+k+1} \equiv b_{n+k} \bmod 6$ but $b_{n+k+1} \not\equiv b_{n+k} \bmod 60$.

Table 1 shows the possible values of (u_n, v_n) for $d_n = u_n - v_n \in [-19, -14] \cup [24, 29]$. Observe that for these values of d_n, the possible values of (u_n, v_n) are distinct mod 6. For these values it follows that when a_j, b_j change by a multiple of 6 that is not a multiple of 60, a_{j+1}, b_{j+1} no longer satisfy $3s_{j+1} = 2t_{j+1}$. Then $\Delta_{j+1} \neq 0$, so $d_{j+2} \neq d_{j+1}$. It follows that (d_n) cannot be eventually constant with constant value in $[-19, -14] \cup [24, 29]$. We've shown that d_n can only move towards 0 from these values, so it further follows that (d_n)

d_n	(u_n, v_n)
29	$(29, 0)$
28	$(28, 0), (29, 1)$
27	$(27, 0), (28, 1), (29, 2)$
26	$(26, 0), (27, 1), (28, 2), (29, 3)$
25	$(25, 0), (26, 1), (27, 2), (28, 3), (29, 4)$
24	$(24, 0), (25, 1), (26, 2), (27, 3), (28, 4), (29, 5)$
-19	$(0, 19)$
-18	$(0, 18), (1, 19)$
-17	$(0, 17), (1, 18), (2, 19)$
-16	$(0, 16), (1, 17), (2, 18), (3, 19)$
-15	$(0, 15), (1, 16), (2, 17), (3, 18), (4, 19)$
-14	$(0, 14), (1, 15), (2, 16), (3, 17), (4, 18), (5, 19)$

Table 1: The possible values of (u_n, v_n) for $d_n = u_n - v_n \in [-19, -14] \cup [24, 29]$.

must eventually enter the interval $[-13, 23]$. The lemma shows that $R_{-13} \cap L_{23}$ is an invariant set under f, so this completes the proof of part (a).

Comments.

Our results show that if (d_n) is not eventually trapped in $[-10, 20]$ then it must be eventually constant, with constant value in $[-13, -11] \cup [21, 23]$. For d_n in $[-13, -11] \cup [21, 23]$ there are pairs of values for (u_n, v_n) that are equal mod 6; for example, for $d_n = 23$ we have the pairs $(23, 0)$ and $(29, 6)$. To shorten the trapping interval via the kinds of methods used here it would be necessary to show that (a_n, b_n) cannot keep moving back and forth between such pairs.

In the numerical experiments we've conducted to date, the sequence (d_n) has always ended up inside the interval $[-10, 20]$. Once inside this interval the sequence has the appearence of a random walk, although of course the underlying sequence (a_n, b_n) is completely deterministic. For $d_n \in [-9, 19]$ it can be checked that there exist pairs (a_n, b_n) realising $d_n = a_n - b_n$ with $d_{n+1} > d_n$, as well as pairs (a_n, b_n) realising $d_n = a_n - b_n$ with $d_{n+1} < d_n$. This suggests that it would not be possible to shorten the trapping interval beyond $[-10, 20]$.

Given positive integers α, β, γ with α, β, γ we may consider positive integer sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ satisfying

$$a_{n+1} = a_n + \alpha \cdot \left\lfloor \frac{b_n}{\alpha\gamma} \right\rfloor, \qquad b_{n+1} = b_n + \beta \cdot \left\lfloor \frac{a_n}{\beta\gamma} \right\rfloor$$

(so problem B4 is the case $\alpha = 2$, $\beta = 3$, $\gamma = 10$). For such sequences the arguments used above go through unchanged, with the constants involved suitably re-expressed in terms of α, β, γ. For the boundary values the substitutions are

$$30 \mapsto \beta\gamma, \qquad -20 \mapsto -\alpha\gamma,$$
$$20 \mapsto (\beta - 1)\gamma, \qquad -10 \mapsto -(\alpha - 1)\gamma,$$

$$24 \mapsto \beta(\gamma-\alpha), \qquad\qquad -14 \mapsto -\alpha(\gamma-\beta).$$

Thus for such sequences, if $d_n = a_n - b_n$ is not eventually trapped in $[-(\alpha-1)\gamma, (\beta-1)\gamma]$ then it must be eventually constant, with constant value in one of the two intervals $(-\alpha\gamma, -(\alpha-1)\gamma)$ and $((\beta-1)\gamma, \beta\gamma)$; and moreover, (d_n) cannot be eventually constant with constant value in $(-\alpha\gamma, -\alpha(\gamma-\beta)]$ or $[\beta(\gamma-\alpha), \beta\gamma)$. Notice however that when $\gamma \le \alpha\beta + 1$ we have

$$(-\alpha\gamma, -(\alpha-1)\gamma) \subseteq (-\alpha\gamma, -\alpha(\gamma-\beta)] \qquad \text{and} \qquad ((\beta-1)\gamma, \beta\gamma) \subseteq [\beta(\gamma-\alpha), \beta\gamma).$$

Thus when $\gamma > \alpha\beta + 1$ (as is the case in the problem) we find that (d_n) is eventually trapped in $(-\alpha(\gamma-\beta), \beta(\gamma-\alpha))$; while for $\gamma \le \alpha\beta + 1$ we get the stronger (and likely best possible) result that (d_n) is eventually trapped in $[-(\beta-1)\gamma, (\alpha-1)\gamma]$.

For the generalisation we assume $a_0 b_0 \ge \alpha\beta\gamma^2$, which implies (a_n) and (b_n) are both eventually strictly increasing. If it is possible for (d_n) to be eventually constant with $d_n \ne 0$ then that would answer part (b) in the negative. If not, then to answer part (b) it's necessary to understand the dynamics of (d_n) once it's trapped in the interval $[-(\alpha-1)\gamma, (\beta-1)\gamma]$.

Problem C1

Let A and B be two fixed positive real numbers. The function f is defined by

$$f(x,y) = \min\left\{x,\ \frac{A}{y},\ y + \frac{B}{x}\right\},$$

for all pairs (x, y) of positive real numbers.

Determine the largest possible value of $f(x, y)$.

Solution

We show that the largest possible value of $f(x, y)$ is $\sqrt{A+B}$.

Solution 1.

Let us first determine the pair (x, y) for which $x = \frac{A}{y} = y + \frac{B}{x}$. Solving this gives $x = \sqrt{A+B}$ and $y = \frac{A}{\sqrt{A+B}}$. Clearly, in this case, $f(x, y) = \sqrt{A+B}$, and we claim that this is indeed also the maximum value of $f(x, y)$. The first quadrant can be divided into three (non-disjoint) regions:

I. $0 < x \le \sqrt{A+B}$ and $y > 0$. In this case it is clear that $f(x, y) \le x \le \sqrt{A+B}$.

II. $x > 0$ and $y \ge \frac{A}{\sqrt{A+B}}$. Here, $f(x, y) \le \frac{A}{y} \le \frac{A}{\frac{A}{\sqrt{A+B}}} = \sqrt{A+B}$.

III. $x \ge \sqrt{A+B}$ and $0 < y \le \frac{A}{\sqrt{A+B}}$. Here we have $f(x, y) \le y + \frac{B}{x} \le \frac{A}{\sqrt{A+B}} + \frac{B}{\sqrt{A+B}} = \sqrt{A+B}$.

Our claim is therefore valid.

Solution 2.

Let $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be the function given by letting $y = \frac{A}{z}$, so that

$$g(x,z) = f\left(x,\ \frac{A}{z}\right) = \min\left(x,\ z,\ \frac{A}{z} + \frac{B}{x}\right).$$

Then g has the same range as f, because $z \mapsto \frac{A}{z}$ is a bijecton $\mathbb{R}^+ \to \mathbb{R}^+$. Now observe that if $x \le z$ then $g(x, z) \le \min\left(x,\ \frac{A}{x} + \frac{B}{x}\right) = g(x, x)$, while if $x \ge z$ then $g(x, z) \le \min\left(z,\ \frac{A}{z} + \frac{B}{z}\right) = g(z, z)$.

So we wish to find the maximum of the function $h(x) = g(x, x) = \min\left(x,\ \frac{A}{x} + \frac{B}{x}\right)$. This must occur when $x = \frac{A+B}{x}$, since x is increasing while $\frac{A+B}{x}$ is decreasing. This gives $x = \sqrt{A+B}$, and we conclude that $\max f(x) = \sqrt{A+B}$.

Solution 3.

We use the fact that $\min(u, v) \le \sqrt{uv}$ for all $u, v \in \mathbb{R}^+$ to reduce the problem to just two cases. If $x \le \frac{A}{y}$, then

$$f(x,y) = \min\left(x,\ y + \frac{B}{x}\right) \le \min\left(x,\ \frac{A}{x} + \frac{B}{x}\right)$$

$$\leq \sqrt{x\left(\frac{A}{x}+\frac{B}{x}\right)} = \sqrt{A+B};$$

while if $x \geq \frac{A}{y}$, then

$$\begin{aligned} f(x,y) = \min\left(\frac{A}{y},\, y+\frac{B}{x}\right) &\leq \min\left(\frac{A}{y},\, y+\frac{By}{A}\right) \\ &\leq \sqrt{\frac{A}{y}\left(y+\frac{By}{A}\right)} = \sqrt{A+B}. \end{aligned}$$

Hence $f(x,y) \leq \sqrt{A+B}$ for all x,y. Moreover, as seen above taking $x = \sqrt{A+B}$, $y = \frac{A}{\sqrt{A+B}}$ we obtain $f(x,y) = \sqrt{A+B}$, so this is the largest possible value of f.

Problem C2

Determine whether the series
$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\lceil \sin n \rceil}}$$
is convergent or divergent.

Here $\lceil x \rceil$ denotes the least integer greater than or equal to x.

Solution

We show that the series is divergent.

Solution 1.

For all $n \geq 1$ the integer $\lceil (2n-1)\pi \rceil$ is the smallest integer greater than $(2n-1)\pi$. It therefore belongs to the interval $(2n-1)\pi, 2n\pi)$, because this interval necessarily contains an integer since it has length $\pi > 1$. It follows that $\sin(\lceil (2n-1)\pi \rceil) < 0$ for all $n \geq 1$, because sin is negative on this interval.

Consider the subsequence $n_k = \lceil (2k-1)\pi \rceil \ \ (k \geq 1)$ of the sequence $1, 2, 3, \ldots$. Then we have $1 + \lceil \sin n_k \rceil = 1 + 0 = 1$ for all $k \geq 1$. Furthermore,
$$n_k = \lceil (2k-1)\pi \rceil < (2k-1)\pi + (2k-1) = (2k-1)(\pi+1) < 2k(\pi+1).$$
Hence, $\frac{1}{n_k} > \frac{1}{2(\pi+1)} \cdot \frac{1}{k}$, and since $\sum_{k=1}^{\infty} \frac{1}{2(\pi+1)} \cdot \frac{1}{k}$ is divergent, we have that
$$\sum_{k=1}^{\infty} \frac{1}{n_k} = \sum_{k=1}^{\infty} \frac{1}{n_k^{1+\lceil \sin n_k \rceil}}$$
is divergent. But since this is a subseries of the original one in question (with all terms positive), the original series is also divergent.

Solution 2.

For each positive integer m consider the integers $5(m-1)+i$ for $1 \leq i \leq 5$. These five integers are equally spaced in an interval of length $4 > \pi$, so cannot all lie in an interval of the form $[2k\pi, (2k+1)\pi]$. It follows that $\lceil \sin(5(m-1)+i) \rceil = 0$ for at least one $1 \leq i \leq 5$.

Write
$$a_n = \frac{1}{n^{1+\lceil \sin n \rceil}}, \qquad b_m = \sum_{i=1}^{5} a_{5(m-1)+i}, \qquad c_m = \frac{1}{5m}.$$

Then the previous paragraph implies $b_m > c_m > 0$ for all m. By the comparison test it follows that $\sum_{m=1}^{\infty} b_m$ diverges, because $\sum_{m=1}^{\infty} c_m$ is divergent. Then since all terms a_n are positive we have
$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\lceil \sin n \rceil}} = \sum_{n=1}^{\infty} a_n = \sum_{m=1}^{\infty} \sum_{i=1}^{5} a_{5(m-1)+i} = \sum_{m=1}^{\infty} b_m,$$
so our original series diverges too.

Problem C3

A grasshopper is sitting on the number line. Initially, it is sitting at the number 0. Each second it jumps one unit to the left or to the right, with equal probability. The directions of the jumps are chosen independently of each other.

Let p denote the probability that, after 2022 jumps, the grasshopper is sitting at a number divisible by 5. Determine whether $p < \frac{1}{5}$, $p = \frac{1}{5}$, or $p > \frac{1}{5}$.

Solution

Solution via a recurrence relation

For $n \geq 0$ let

$$\begin{aligned}
a_n &= \text{ the probability that the grasshopper sits in } 5\mathbb{Z} \text{ after } n \text{ jumps,}\\
b_n &= \text{ the probability that the grasshopper sits in } 5\mathbb{Z}+1 \text{ after } n \text{ jumps,}\\
c_n &= \text{ the probability that the grasshopper sits in } 5\mathbb{Z}+2 \text{ after } n \text{ jumps,}\\
d_n &= \text{ the probability that the grasshopper sits in } 5\mathbb{Z}+3 \text{ after } n \text{ jumps,}\\
e_n &= \text{ the probability that the grasshopper sits in } 5\mathbb{Z}+4 \text{ after } n \text{ jumps.}
\end{aligned}$$

Then, for all $n \geq 0$,

$$\begin{aligned}
a_{n+1} &= \tfrac{1}{2}b_n + \tfrac{1}{2}e_n\\
b_{n+1} &= \tfrac{1}{2}a_n + \tfrac{1}{2}c_n\\
c_{n+1} &= \tfrac{1}{2}b_n + \tfrac{1}{2}d_n\\
d_{n+1} &= \tfrac{1}{2}c_n + \tfrac{1}{2}e_n\\
e_{n+1} &= \tfrac{1}{2}d_n + \tfrac{1}{2}a_n.
\end{aligned}$$

We also have $a_0 = 1;\ b_0 = c_0 = d_0 = e_0 = 0$ and $a_1 = 0;\ b_1 = \frac{1}{2};\ c_1 = 0;\ d_1 = 0;\ e_1 = \frac{1}{2}$.

We claim that $b_n = e_n$ and $c_n = d_n$ for all $n \geq 0$. This is readily proved by induction, but also follows from a symmetry argument. Since the grasshopper jumps left or right with equal probabilities, the probability that it is sitting at the integer k after n jumps is equal to the probability that it is sitting at the integer $-k$. As sets we have $(5\mathbb{Z}+1) = -(5\mathbb{Z}+4)$ and $(5\mathbb{Z}+2) = -(5\mathbb{Z}+3)$, from which the claim follows. It follows that

$$\begin{aligned}
a_{n+1} &= b_n\\
b_{n+1} &= \tfrac{1}{2}a_n + \tfrac{1}{2}c_n\\
c_{n+1} &= \tfrac{1}{2}b_n + \tfrac{1}{2}c_n,
\end{aligned}$$

for all $n \geq 0$.

We now present two ways to finish the problem from here. For the first, we will show that $p > \frac{1}{5}$ without explicitly solving the recurrence relation, by proving by induction that $a_n > c_n > b_n$ for all even $n > 0$. In the base case $n = 2$ we have $a_2 = \frac{1}{2}$, $c_2 = \frac{1}{4}$ and $b_2 = 0$, so the claim holds. For the inductive step we use the relations above to obtain

$$a_{n+2} = \tfrac{1}{2}a_n + \tfrac{1}{2}c_n,$$

$$\begin{aligned} b_{n+2} &= \tfrac{3}{4}b_n + \tfrac{1}{4}c_n, \\ c_{n+2} &= \tfrac{1}{4}a_n + \tfrac{1}{4}b_n + \tfrac{1}{2}c_n, \end{aligned}$$

for all $n \geq 0$. Supposing now that $a_n > c_n > b_n$ for some $n \geq 0$ we have

$$\begin{aligned} a_{n+2} - c_{n+2} &= \tfrac{1}{4}(a_n - b_n) > 0, \\ c_{n+2} - b_{n+2} &= \tfrac{1}{4}(a_n - b_n) + \tfrac{1}{4}(c_n - b_n) > 0, \end{aligned}$$

completing the inductive step. To finish the problem we observe that for all even $n > 0$, since a_n, b_n, c_n, d_n, e_n sum to 1 and are not all equal the largest, namely a_n, must be strictly greater than the average $\frac{1}{5}$.

For the second approach we will use the relations above to obtain an explicit formula for a_n. Combining the three relations we find that, for $n \geq 2$,

$$\begin{aligned} a_{n+1} &= \tfrac{1}{2}(a_{n-1} + c_{n-1}) \\ &= \tfrac{1}{2}(a_{n-1} + \tfrac{1}{2}(b_{n-2} + c_{n-2})) \\ &= \tfrac{1}{2}(a_{n-1} + \tfrac{1}{2}b_{n-2} + b_{n-1} - \tfrac{1}{2}a_{n-2}) \\ &= \tfrac{1}{2}(a_{n-1} + \tfrac{1}{2}a_{n-1} + a_n - \tfrac{1}{2}a_{n-2}) \\ &= \tfrac{1}{2}a_n + \tfrac{3}{4}a_{n-1} - \tfrac{1}{4}a_{n-2}. \end{aligned}$$

The characteristic equation of this recurrence relation is $x^3 - \frac{1}{2}x^2 - \frac{3}{4}x + \frac{1}{4} = 0$. An obvious root is $x = 1$, and the other two roots follow easily: $x = \frac{1}{4}(-1 \pm \sqrt{5})$. So there exist constants A, B, C such that $a_n = A + B(\frac{1}{4}(-1+\sqrt{5}))^n + C(\frac{1}{4}(-1-\sqrt{5}))^n$ for all $n \geq 0$. Using $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{1}{2}$, we easily find the values of these constants: $A = \frac{1}{5}$, $B = \frac{2}{5}$, $C = \frac{2}{5}$. Consequently, for all $n \geq 0$,

$$a_n = \tfrac{1}{5} + \tfrac{2}{5}\left[(\tfrac{1}{4})^n\left((-1+\sqrt{5})^n + (-1-\sqrt{5})^n\right)\right].$$

Since the expression between the square brackets is positive for even n, it follows that $a_n > \frac{1}{5}$ if n is even. In particular, $p = a_{2022} > \frac{1}{5}$.

Solution via a generating function

For each integer k and nonnegative integer n let $g_n(k)$ be the probability that the grasshopper is sitting at the number k after n jumps, and let

$$G_n(x) = \sum_{k\in\mathbb{Z}} g_n(k)x^k$$

be the generating function of the sequence $(g_n(k))_{k\in\mathbb{Z}}$. For all $n \geq 0$ and all $k \in \mathbb{Z}$ we have

$$g_{n+1}(k) = \frac{g_n(k-1) + g_n(k+1)}{2},$$

and it follows that

$$G_{n+1}(x) = \tfrac{1}{2}(x + x^{-1})G_n(x).$$

Since $G_0(x) = 1$ we obtain

$$G_n(x) = \left(\frac{x + x^{-1}}{2}\right)^n.$$

Note that G_n is a polynomial in x and x^{-1} for all n.

We are interested in the sum $p = \sum_{k \in 5\mathbb{Z}} g_{2022}(k)$. More generally let $p_n = \sum_{k \in 5\mathbb{Z}} g_n(k)$, and let $\zeta = e^{2\pi i/5}$ be a fifth root of unity. We claim that

$$\sum_{j=0}^{4} G_n(\zeta^j) = 5p_n.$$

Indeed,

$$\begin{aligned} \sum_{j=0}^{4} G_n(\zeta^j) &= \sum_{j=0}^{4} \sum_{k \in \mathbb{Z}} g_n(k)\zeta^{jk} \\ &= \sum_{k \in \mathbb{Z}} g_n(k) \sum_{j=0}^{4} \zeta^{jk}. \end{aligned}$$

If $k \in 5\mathbb{Z}$ then $\zeta^{jk} = 1$ for all j, and otherwise $\zeta^0, \zeta^k, \zeta^{2k}, \zeta^{3k}, \zeta^{4k}$ are equal to $1, \zeta^1, \zeta^2, \zeta^3, \zeta^4$ in some order. Since $\sum_{j=0}^{4} \zeta^j = 0$ (because $(\zeta - 1)\sum_{j=0}^{4} \zeta^j = \zeta^5 - 1 = 0$) the claim follows.

Now

$$\begin{aligned} G_n(1) &= 1, \\ G_n(\zeta) = G_n(\zeta^4) &= \cos^n \frac{2\pi}{5}, \\ G_n(\zeta^2) = G_n(\zeta^3) &= \cos^n \frac{4\pi}{5}, \end{aligned}$$

so

$$p_n = \frac{1}{5}\left(1 + 2\cos^n \frac{2\pi}{5} + 2\cos^n \frac{4\pi}{5}\right).$$

Finally when $n = 2m$ is even we have

$$5p_n = 1 + 2\left(\cos^m \frac{2\pi}{5}\right)^2 + 2\left(\cos^m \frac{4\pi}{5}\right)^2 > 1,$$

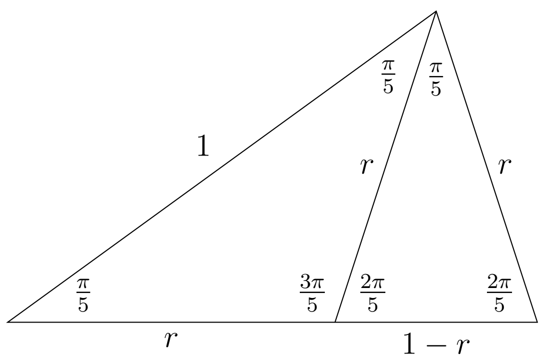
so $p = p_{2022} > \frac{1}{5}$.

Comment

Using the triangle below it can be shown that the length r satisfies $r^2 + r - 1 = 0$, so that $r = \frac{-1+\sqrt{5}}{2}$. Then

$$\begin{aligned} \cos \frac{2\pi}{5} &= \frac{r}{2} = \frac{-1+\sqrt{5}}{4}, \\ \cos \frac{4\pi}{5} &= 1 - 2\cos^2 \frac{2\pi}{5} = \frac{-1-\sqrt{5}}{4}, \end{aligned}$$

so our expression for p_n agrees with our expression for a_n, as it should.


$\frac{\pi}{5}$
$\frac{\pi}{5}$
1
r
r
$\frac{\pi}{5}$
$\frac{3\pi}{5}$
$\frac{2\pi}{5}$
$\frac{2\pi}{5}$
r
$1-r$

Problem C4

A machine is programmed to output a sequence of positive integers $a_1, a_2, a_3, \ldots$. It outputs the integers independently, one at a time, and at each step the probability that it outputs the integer k is equal to $\frac{1}{2^k}$. The machine stops when it outputs an integer that it has already output before.

Prove that the probability that the machine stops when it has output exactly n integers is

$$\frac{(n-1)!\,(2^n - n - 1)}{\prod_{r=1}^{n}(2^r - 1)}.$$

Solution

For convenience we will assume that the machine doesn't stop when it has output an integer it has output before, and instead outputs an integer a_m for all $m \geq 1$. Then we wish to find the probability that n is the least integer such that $a_n = a_k$ for some $k < n$.

For $n \geq 1$ let A_n be the event that n is the least integer for which $a_n = a_k$ for some $k < n$, and let B_n be the event that $a_1, a_2, \ldots, a_n$ are all different. Thus, we wish to find $P(A_n)$. However, the event A_n is exactly the event that B_{n-1} occurs but B_n does not. Since $B_n \subseteq B_{n-1}$ for all n we therefore have

$$P(A_n) = P(B_{n-1} - B_n) = P(B_{n-1}) - P(B_n).$$

This shows that if we can find a general expression for the probability that the first n terms in the sequence are all different, then we can compute $P(A_n)$.

The probability that the first n integers are distinct is given by the sum

$$\sum_{\substack{(a_1,a_2,\ldots,a_n)\in(\mathbb{Z}^+)^n \\ |\{a_1,\ldots,a_n\}|=n}} \frac{1}{2^{a_1}} \times \frac{1}{2^{a_2}} \times \cdots \times \frac{1}{2^{a_n}};$$

but if we let X_n be the set of n-tuples of integers given by

$$X_n = \{(a_1, a_2, \ldots, a_n) : 0 < a_1 < a_2 < \cdots < a_n\}$$

then we can order the a_i from the sum above and compensate by a factor of $n!$ to get

$$P(B_n) = n!\left(\sum_{(a_1,\ldots,a_n)\in X_n} \frac{1}{2^{a_1}} \times \frac{1}{2^{a_2}} \times \cdots \times \frac{1}{2^{a_n}}\right).$$

Let the expression in parentheses above be equal to $T(n)$. We will show how to compute $T(n)$ inductively. First, we define Y_n to be the set of n-tuples of integers

$$Y_n = \{(a_1, a_2, \ldots, a_n) : 1 < a_1 < a_2 < \cdots < a_n\},$$

and observe that

$$(a_1, a_2, \ldots, a_n) \mapsto (a_1 + 1, a_2 + 1, \ldots, a_n + 1)$$

is a bijection from X_n to Y_n. This means

$$\begin{aligned}
T(n) &= \sum_{(a_1,\ldots,a_n)\in X_n} \frac{1}{2^{a_1}} \times \cdots \times \frac{1}{2^{a_n}} \\
&= \left(\sum_{\substack{(a_1,\ldots,a_n)\in X_n \\ a_1=1}} \frac{1}{2^{a_1}} \times \cdots \times \frac{1}{2^{a_n}} \right) + \left(\sum_{\substack{(a_1,\ldots,a_n)\in X_n \\ a_1>1}} \frac{1}{2^{a_1}} \times \cdots \times \frac{1}{2^{a_n}} \right) \\
&= \frac{1}{2} \left(\sum_{(a_2,\ldots,a_n)\in Y_{n-1}} \frac{1}{2^{a_2}} \times \cdots \times \frac{1}{2^{a_n}} \right) + \left(\sum_{(a_1,\ldots,a_n)\in Y_n} \frac{1}{2^{a_1}} \times \cdots \times \frac{1}{2^{a_n}} \right) \\
&= \frac{1}{2} \times \frac{1}{2^{n-1}} \left(\sum_{(a_2,\ldots,a_n)\in X_{n-1}} \frac{1}{2^{a_2}} \times \cdots \times \frac{1}{2^{a_n}} \right) + \frac{1}{2^n} \left(\sum_{(a_1,\ldots,a_n)\in X_n} \frac{1}{2^{a_1}} \times \cdots \times \frac{1}{2^{a_n}} \right) \\
&= \frac{1}{2^n} T(n-1) + \frac{1}{2^n} T(n),
\end{aligned}$$

which can be rearranged to get $T(n) = \dfrac{1}{2^n - 1} T(n-1)$. It is easily checked that $T(1) = 1$, and then induction can be used to show that

$$T(n) = \prod_{r=1}^{n} \frac{1}{2^r - 1}$$

for all n. Hence

$$P(B_n) = n! T(n) = \prod_{r=1}^{n} \frac{r}{2^r - 1},$$

from which we can compute $P(A_n)$ as

$$\begin{aligned}
P(A_n) &= P(B_{n-1}) - P(B_n) \\
&= \prod_{r=1}^{n-1} \frac{r}{2^r - 1} - \prod_{r=1}^{n} \frac{r}{2^r - 1} \\
&= \left(1 - \frac{n}{2^n - 1} \right) \prod_{r=1}^{n-1} \frac{r}{2^r - 1} \\
&= \left(\frac{2^n - 1 - n}{2^n - 1} \right) \prod_{r=1}^{n-1} \frac{r}{2^r - 1} \\
&= \frac{(n-1)!(2^n - n - 1)}{\displaystyle\prod_{r=1}^{n} (2^r - 1)},
\end{aligned}$$

as required.

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