2023 PAPER C: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem is worth 7 points.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions. Tablets may be used solely for writing worked solutions, with internet access switched off.
- For participants sitting the exam off-site, an electronic device such as a PC, laptop, phone or tablet may be used during the competition for accessing the papers, undergoing invigilation, writing and submitting solutions and (for pairs entrants) communicating with the other member of the pair.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "C3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and the paper (as in **S1234567C** (for singles) or **P3141593C** (for pairs)), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

SIMON MARAIS mathematics competition

2023 PAPER C: PROBLEMS

C1. There are 2023 cups numbered from 1 through 2023. Red, green, and blue balls are placed in the cups according to the following rules.

- If cups m and n both contain a red ball, then $m-n$ is a multiple of 2.
- If cups m and n both contain a green ball, then $m-n$ is a multiple of 3.
- If cups m and n both contain a blue ball, then $m-n$ is a multiple of 5.

What is the smallest possible number of empty cups?

C2. For an integer $n \geq 2$, consider the line segment connecting the point $(0,k)$ to the point $(n-k,0)$ for $k = 0,1,2,\ldots,n$. The union of these $n+1$ line segments divides the plane into one unbounded region and a number of bounded regions, each of which is a triangle or a quadrilateral. Each of these bounded regions can be coloured blue or red in a unique way such that regions sharing an edge have different colours and the region with vertex $(0,0)$ is coloured blue.

Determine all values of n for which the total area that is coloured blue is equal to the total area that is coloured red.

C3. Determine the maximum real number C such that

$$\sum_{i=1}^{n} \frac{x_i}{x_{i-1}} \geq n + C,$$

for all positive integers n and all sequences of positive real numbers $x_0, x_1, \ldots, x_n$ such that $x_0 = 1$ and $x_n = 2$.

C4. Let k be a positive integer. Keira and Roland play a game of *reverse chess*. Initially, Roland chooses a positive integer $n > \frac{k}{2023}$. Keira places k kings on k distinct squares of a $2023 \times n$ chess board. Then Roland places a rook on an unoccupied square of the board. Both players then take turns moving any number (possibly zero) of their pieces, with Keira starting first. Each king cannot move to a square occupied by another king, but it can capture the rook. Furthermore, if Keira chooses to move more than one king in her turn, she moves them one at a time. Roland's rook is not permitted to capture any king, nor may it pass through a square occupied by a king.

For which k can Keira guarantee to capture Roland's rook, regardless of Roland's moves or choice of n?

(A king can move exactly one square in any horizontal, vertical or diagonal direction. A rook can move any number of squares in a horizontal or vertical direction. One piece (rook or king) captures another by moving to the square it occupies.)

