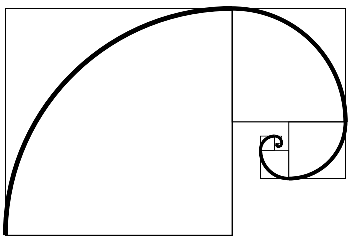
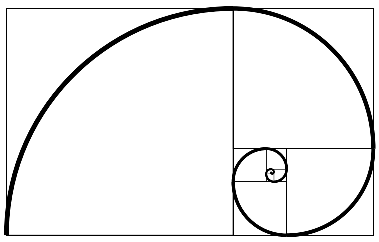
2023 SOLUTIONS

Problem A1

For $0 < q < 1$, the *q-Fibonacci spiral* is constructed as described below. An arc of radius 1 is first drawn inside a 1×1 square. A second arc is then drawn in a $q \times q$ square, then a third in a $q^2 \times q^2$ square, and so on ad infinitum, to create a continuous curve. The example on the left shows the case $q = \frac{1}{2}$, while the example on the right shows $q = \frac{\sqrt{5}-1}{2}$.



$q = \frac{1}{2}$



$q = \frac{\sqrt{5}-1}{2}$

Prove that there exists a circle centred at the centre of the initial 1×1 square such that for each $0 < q < 1$, the limiting endpoint of the q-Fibonacci spiral lies on this circle.

Solution 1

Let $(0, 0)$ be the coordinates of the bottom left corner of the 1×1 square and let (x, y) be the coordinates of the other end of the spiral. The values of x and y can be found by tracking the positive and negative contributions of the radii for each successive arc. These radii form a geometric sequence with common ratio equal to the scale factor q. Hence we have

$$\begin{aligned} x &= 1 + q - q^2 - q^3 + q^4 + q^5 - q^6 - q^7 + \cdots \\ &= (1 + q)\left(1 - q^2 + q^4 - q^6 + \cdots\right) \\ &= \frac{1 + q}{1 + q^2} \end{aligned}$$

and

$$\begin{aligned} y &= 1 - q - q^2 + q^3 + q^4 - q^5 - q^6 + q^7 + \cdots \\ &= (1 - q)\left(1 - q^2 + q^4 - q^6 + \cdots\right) \end{aligned}$$

$$= \frac{1-q}{1+q^2},$$

twice using the formula for an infinite geometric series $1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$.

The centre of the initial square is at $(\frac{1}{2}, \frac{1}{2})$. It remains to show that the distance from the point $(\frac{1+q}{1+q^2}, \frac{1-q}{1+q^2})$ to $(\frac{1}{2}, \frac{1}{2})$ is independent of q. We compute this distance is

$$\sqrt{\left(\frac{1+q}{1+q^2} - \frac{1}{2}\right)^2 + \left(\frac{1-q}{1+q^2} - \frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}.$$

Solution 2

We work in the complex plane. Let z_0 be the start of the spiral, and for $n \geq 1$, let z_n be the point where the spiral exits the n-th square.

The consecutive diagonals $z_{n+1} - z_n$ and $z_{n-1} - z_n$ are at right angles and their lengths differ by a multiple of q. So we obtain the equation

$$z_{n+1} - z_n = iq(z_{n-1} - z_n)$$

for all $n \geq 1$.

This is a linear recurrence relation with characteristic equation

$$\lambda^2 - \lambda = iq(1 - \lambda)$$

which has roots $\lambda = 1, -iq$.

Therefore the solution is of the form

$$z_n = A + B(-iq)^n$$

for some $A, B \in \mathbb{C}$. From $z_0 = A + B$ and $z_1 = A - iqB$ we obtain

$$B = \frac{z_0 - z_1}{1 + iq}.$$

The centre of the initial square is

$$\frac{z_0 + z_1}{2} = A + B\frac{1 - iq}{2}$$

and the limiting endpoint of the q-Fibonnaci spiral is

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (A + B(-iq)^n) = A.$$

The distance between these two points is

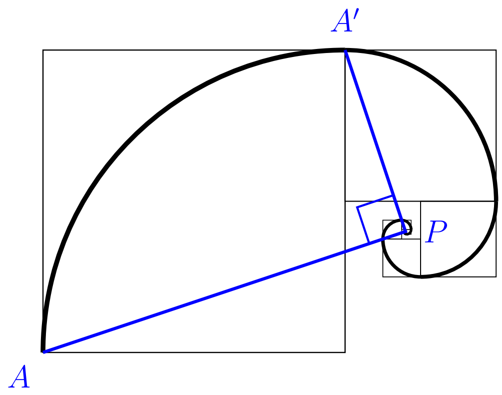
$$\left|\left(A + B\frac{1-iq}{2}\right) - A\right| = |B| \left(\frac{1-iq}{2}\right) = \frac{|z_0 - z_1|}{2} \frac{|1-iq|}{|1+iq|}.$$

Since $q \in \mathbb{R}$, $|1 - iq| = |1 + iq|$ and therefore this distance is independent of q, as required.

Solution 3

Notice that the full spiral can be scaled by a factor of q and rotated $90°$ clockwise to create the same spiral without the first 1×1 square. The centre of this spiral similarity is the point that remains the same under the transformation, namely the limiting endpoint of the spiral, labelled P below.

Since the bottom-left corner of the 1×1 square (A) is moved to the top-right corner (A') under this transformation, the angle APA' must be $90°$. So P must lie on the circumference of the circle with diameter AA'.



Problem A2

Let n be a positive integer and let $f_1(x), \dots, f_n(x)$ be affine functions from $\mathbb{R}$ to $\mathbb{R}$ such that, amongst the n graphs of these functions, no two are parallel and no three are concurrent. Let S be the set of all convex functions $g(x)$ from $\mathbb{R}$ to $\mathbb{R}$ such that for each $x \in \mathbb{R}$, there exists i such that $g(x) = f_i(x)$.

Determine the largest and smallest possible values of $|S|$ in terms of n.

(A function $f(x)$ is *affine* if it is of the form $f(x) = ax + b$ for some $a, b \in \mathbb{R}$. A function $g(x)$ is *convex* if $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y)$ for all $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$.)

Solution

We will prove that largest possible value of $|S|$ is $2^n - 1$ and the smallest possible value is $\frac{n(n+1)}{2}$.

Suppose $g \in S$. For $i = 1, 2, \dots, n$, let $X_i = \{x \in \mathbb{R} : g(x) = f_i(x)\}$. We will first note the following.

Claim: If $|X_i| \geq 3$, then X_i is an interval.

This claim follows since if $a < b < c$ are such that $(a, g(a))$, $(b, g(b))$, and $(c, g(c))$ are collinear, then since g is convex, g is linear on $[a, b]$. Apply this observation where $a, b, c \in X_i$.

Let $T = \{i : |X_i| \geq 3\}$. We will prove that

$$g(x) = \max_{i \in T} f_i(x). \tag{1}$$

We first note that the claim above, along with the fact that $g(x)$ is convex, implies that $g(x) \geq f_i(x)$ for all $i \in T$, and therefore $g(x) \geq \max_{i \in T} f_i(x)$.

Suppose that there exists $y \in \mathbb{R}$ with $g(y) \neq \max_{i \in T} f_i(y)$. Then $g(y) > f_i(y)$ for all $i \in T$. Since convex functions are continuous, there exists an open neighbourhood U of y such that for all $z \in U$, $g(z) > f_i(z)$ for all $z \in U$. For each $z \in U$, there must exist j with $f_j(z) = g(z)$. Since the set of possible j is finite and U is infinite, there exists j such that $f_j(z) = g(z)$ for at least three $z \in U$. But therefore $j \in T$ which implies $g(z) > f_j(z)$, a contradiction. This completes the proof of (1).

Since T is nonempty, there are at most $2^n - 1$ possible subsets T of $\{1, 2, \dots, n\}$. This upper bound can be achieved, for example we can take the graphs of $f_1, \dots, f_n$ to be n distinct tangents of the graph $y = x^2$. In this case every subset T gives a distinct function because for every i, there exists some x such that $f_i(x) = \max_j f_j(x)$.

For the lower bound, since no two lines are parallel, all functions f_i with $1 \leq i \leq n$ and $\max(f_i, f_j)$ with $1 \leq i < j \leq n$ are pairwise distinct elements of S. So there are at least $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ such functions. To achieve this lower bound, we can take n distinct tangents of the graph $y = -x^2$. If we order the functions in increasing gradients, then it is easy to show that $f_j < \max(f_i, f_k)$ for $i < j < k$, so there are no other elements of S.

Problem A3

For each positive integer n, let $f(n)$ denote the smallest possible value of

$$|A_1 \cup A_2 \cup \cdots \cup A_n|,$$

where $A_1, A_2, \ldots, A_n$ are sets such that $A_i \not\subseteq A_j$ and $|A_i| \neq |A_j|$ whenever $i \neq j$.

Determine $f(n)$ for each positive integer n.

Solution

We will prove that

$$f(1) = 0, \qquad f(2) = 3, \qquad \text{and} \qquad f(n) = n + 2 \text{ for } n \geq 3.$$

It is clear that $f(1) = 0$ by taking $A_1 = \{\}$.

We obtain $f(2) \leq 3$ by taking $A_1 = \{1\}$ and $A_2 = \{2, 3\}$. Since in order to satisfy the subset condition, neither set can be empty and a singleton set must contain a unique element, it is clear that $f(2) \geq 1 + 2 = 3$, so we conclude that $f(2) = 3$.

Next, we prove that $f(n) \geq n + 2$ for $n \geq 3$. Suppose that $A_1, A_2, \ldots, A_n$ satisfy the conditions of the problem and are listed in increasing order of cardinality. We first observe that for all $1 \leq k \leq n - 1$ we must have $|A_{k+1}| \geq |A_k| + 1$. We also observe in particular that $A_1 \not\subseteq A_n$, $A_2 \not\subseteq A_n$, and $A_1 \not\subseteq A_2$.

We find that there is never a solution such that $|A_n| < n$, since this would require through our first observation that $|A_1| < 1$. This is impossible, since $A_1 \not\subseteq A_n$.

If $|A_n| = n$, then we observe that A_1 must contain a single element which appears in neither A_2 nor A_n, and A_2 must contain a pair of elements such that at least one does not appear in A_n. Therefore,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \geq |A_1 \cup A_2 \cup A_n| \geq |A_n| + 2 = n + 2.$$

On the other hand, if $|A_n| > n$, then we observe that there must be at least one element in any arbitrarily chosen set A_k for $1 \leq k < n$ which is not in A_n. We select $k = 1$ without loss of generality, and find

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \geq |A_1 \cup A_n| \geq |A_n| + 2 = n + 2.$$

In either case, we find that $f(n) \geq n + 2$.

It remains to show that $f(n) \leq n + 2$ for $n \geq 3$, which we do using an inductive construction. The base cases are given by $n = 3$ and $n = 4$ below, and it is easy to check that they do indeed satisfy the conditions of the problem.

$$\begin{aligned} n = 3: &\quad A_1 = \{1\},\ A_2 = \{2, 3\},\ A_3 = \{3, 4, 5\} \\ n = 4: &\quad A_1 = \{1\},\ A_2 = \{2, 3\},\ A_3 = \{2, 4, 5\},\ A_4 = \{3, 4, 5, 6\} \end{aligned}$$

Now suppose that for some $n \geq 3$, sets $A_1, A_2, \ldots, A_n$ satisfy the following conditions:

- $A_i \not\subseteq A_j$ for $i \neq j$;
- $|A_i| \neq |A_j|$ for $i \neq j$;
- $|A_k| = k$ for $1 \leq k \leq n$; and
- $|A_1 \cup A_2 \cup \cdots \cup A_n| = n + 2$.

If these more restrictive conditions are satisfied, we necessarily have $f(n) \leq n + 2$.

Write $A = A_1 \cup A_2 \cup \cdots \cup A_n$ and let a and b be two elements that do not belong to A. Consider the following $n + 2$ sets:

$$\begin{aligned} B_1 &= \{a\} \\ B_{k+1} &= A_k \cup \{b\}, \quad \text{for} \quad k = 1, 2, \ldots, n \\ B_{n+2} &= A \end{aligned}$$

We now show that the new sets $B_1, B_2, \cdots, B_{n+2}$ satisfy all four of the conditions we require in our base case. First, we prove the subset condition:

- By the inductive assumption, we know B_i is not a subset of B_j for $2 \leq i < j \leq n+1$.
- By construction, B_1 is not a subset of B_j for $2 \leq j \leq n + 2$.
- By construction, B_i is not a subset of B_{n+2} for $1 \leq i \leq n + 1$.

It follows that B_i is not a subset of B_j for any $i \neq j$.

Next, we prove the conditions that $|B_i| \neq |B_j|$ for $i \neq j$ and, more generally, that $|B_k| = k$. We have this by construction for $k = 1$ and by the inductive hypothesis for $2 \leq k \leq n+2$, so both are satisfied.

Lastly, we observe that $|B_1 \cup B_2 \cup \cdots \cup B_{n+2}| = n + 4$, so the final property is satisfied as well.

Combined with the prior base cases for $f(3)$ and $f(4)$, this construction shows by induction that $f(n) \leq n + 2$ for all $n \geq 3$. Since we already know from our previous work that $f(n) \geq n + 2$, we conclude that $f(n) = n + 2$ for $n \geq 3$.

Problem A4

Let $x_0, x_1, x_2, \ldots$ be a sequence of positive real numbers such that for all $n \geq 0$,

$$x_{n+1} = \frac{(n^2+1)x_n^2}{x_n^3+n^2}.$$

For which values of x_0 is this sequence bounded?

Solution

First consider the derivative of the function $f_n(x) = \frac{(n^2+1)x_n^2}{x_n^3+n^2}$.

$$\begin{aligned} f_n'(x) &= \frac{2(n^2+1)x(x^3+n^2) - (n^2+1)x^2(3x^2)}{(x^3+n^2)^2} \\ &= \frac{(n^2+1)x}{(x^3+n^2)^2} \times \left(2n^2 - x^3\right). \end{aligned}$$

So there is a unique critical point at $x = \sqrt[3]{2n^2}$ and so $f_n(x)$ is increasing on the interval $(0, \sqrt[3]{2n^2})$ and decreasing on the interval $(\sqrt[3]{2n^2}, \infty)$.

We now let $x_0 = \lambda$ and find the following first few terms.

$$\begin{aligned} x_0 &= \lambda \\ x_1 &= \frac{(0^2+1)\lambda^2}{\lambda^3+0^2} = \lambda^{-1} \\ x_2 &= \frac{(1^2+1)x_1^2}{x_1^3+1^2} = \frac{2\lambda^{-2}}{\lambda^{-3}+1} = \frac{2\lambda}{1+\lambda^3} \end{aligned}$$

Observation One: For any $n \geq 2$, if $x_n \leq 1$ then $x_{n+1} \leq 1$.
Proof: Since $f_n(x)$ is increasing on the interval $0 < x \leq 1$, we have $f_n(x_n) \leq f_n(1)$. Therefore

$$x_{n+1} = f_n(x_n) \leq f_n(1) = \frac{(n^2+1)1^2}{1^3+n^2} = 1.$$

Observation Two: For any $n \geq 2$, if $1 < x_n < n^2$ then $1 < x_{n+1} < (n+1)^2$.
Proof: Since $f_n(x)$ is increasing on the interval $1 < x < \sqrt[3]{2n^2}$ and decreasing on the interval $(\sqrt[3]{2n^2}, n)$, we simply need to verify that

- $f_n(1) = \frac{(n^2+1)1^2}{1^3+n^2} = 1$, and
- $f_n(n^2) = \frac{(n^2+1)n^4}{n^6+n^2} = \frac{n^4+n^2}{n^4+1} \geq 1$, and
- $f_n(\sqrt[3]{2n^2}) = \frac{(n^2+1)(\sqrt[3]{2n^2})^2}{2n^2+n^2} = \frac{\sqrt[3]{4}}{3n^{2/3}}(n^2+1) < (n+1)^2$.

The last statement follows from the fact that $\frac{\sqrt[3]{4}}{3n^{2/3}} < 1$ and $n^2+1 < (n+1)^2$.

With these two observations out of the way, we turn our attention to the original problem.

If $x_2 \leq 1$, then the sequence is bounded by Observation One. So let's assume that $x_2 > 1$. Note that $x_2 = \frac{2\lambda}{1+\lambda^3} < 4$ for all positive λ, so by Observation Two we can assume that $1 < x_n < n^2$ for all $n \geq 2$. Now for the sake of contradiction assume that there exists a constant b such that

$$1 < x_n < b$$

for all $n \geq 2$. Then:

$$\begin{aligned} x_{n+1} - x_n &= \frac{(n^2+1)x_n^2}{x_n^3+n^2} - x_n \\ &= \frac{-x_n^4 + (n^2+1)x_n^2 - n^2 x_n}{x_n^3+n^2} \\ &= \frac{x_n(x_n-1)(n^2-x_n^2-x_n)}{x_n^3+n^2} \\ &\geq \frac{x_n(x_n-1)(n^2-b^2-b)}{n^2+b^3}. \end{aligned}$$

For all n greater than some large enough value N, we have $(n^2 - 2b^2 - b) > \frac{n^2}{2}$ and $(n^2 + b^3) < 2n^2$. So eventually we get

$$x_{n+1} - x_n \geq \frac{x_n(x_n-1)(n^2-b^2-b)}{n^2+b^3} > \frac{x_n(x_n-1)\frac{n^2}{2}}{2n^2} = \frac{x_n(x_n-1)}{4} > 0,$$

meaning the sequence (x_n) is increasing for $n \geq N$. Letting $a = x_N > 1$, we can conclude that for all $n \geq N$,

$$x_{n+1} - x_n \geq \frac{a(a-1)}{4}.$$

Therefore the sequence is eventually always greater than an arithmetic progression with common difference $d = \frac{a(a-1)}{4}$, which contradicts the assumption that b was an upper bound.

So the sequence is unbounded if and only if $x_2 > 1$.

Solving the inequality $x_2 = \frac{2x_0}{x_0^3+1} > 1$ yields the final answer that the sequence is bounded if and only if $0 < x_0 \leq \frac{\sqrt{5}-1}{2}$ or $x_0 \geq 1$.

Problem B1

Find the smallest positive real number r with the following property: For every choice of 2023 unit vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2023} \in \mathbb{R}^2$, a point $\mathbf{p}$ can be found in the plane such that for each subset S of $\{1, 2, \ldots, 2023\}$, the sum

$$\sum_{i \in S} \mathbf{v}_i$$

lies inside the disc $\{\mathbf{x} \in \mathbb{R}^2 : ||\mathbf{x} - \mathbf{p}|| \leq r\}$.

Solution

We will show that the answer is $r = \frac{2023}{2}$.

Let $\mathbf{p} = \frac{1}{2} \sum\limits_{i=1}^{2023} \mathbf{v}_i$. If $S \subseteq \{1, 2, \ldots, 2023\}$ then

$$\left|\left|\left(\sum_{i \in S} \mathbf{v}_i\right) - \mathbf{p}\right|\right| = \frac{1}{2}\left|\left|\sum_{i \in S} \mathbf{v}_i - \sum_{i \notin S} \mathbf{v}_i\right|\right| \leq \frac{1}{2}\left(\sum_{i \in S} ||\mathbf{v}_i|| + \sum_{i \notin S} ||-\mathbf{v}_i||\right) = \frac{1}{2} \times 2023$$

by the triangle inequality. Therefore the sum $\sum_{i \in S} \mathbf{v}_i$ lies within the disc centred at $\mathbf{p}$ with radius $\frac{2023}{2}$.

On the other hand, if $\mathbf{v}_1 = \mathbf{v}_2 = \cdots = \mathbf{v}_{2023}$ then the sums corresponding to $S = \emptyset$ and $S = \{1, 2, \ldots, 2023\}$ are distance 2023 apart, so cannot both lie in any disc of radius less than $\frac{2023}{2}$. Therefore $\frac{2023}{2}$ is the optimal value of r.

Problem B2

There are 256 players in a tennis tournament who are ranked from 1 to 256, with 1 corresponding to the highest rank and 256 corresponding to the lowest rank. When two players play a match in the tournament, the player whose rank is higher wins the match with probability $\frac{3}{5}$.

In each round of the tournament, the player with the highest rank plays against the player with the second highest rank, the player with the third highest rank plays against the player with the fourth highest rank, and so on. At the end of the round, the players who win proceed to the next round and the players who lose exit the tournament. After eight rounds, there is one player remaining in the tournament and they are declared the winner.

Determine the expected value of the rank of the winner.

Solution 1

More generally, suppose that there are 2^n players, so that the tournament lasts for n rounds. Furthermore, suppose that the person whose rank is higher wins the match with probability p. Let W be the random variable corresponding to the rank of the winner. We will prove by induction that for every positive integer n, the expected value is

$$E[W] = 2^n - 2^n p + p.$$

In the case $n = 1$, the tournament consists of only one match and we have

$$E[W] = 1 \times P(X=1) + 2 \times P(X=2) = 1 \times p + 2 \times (1-p) = 2 - p.$$

Therefore, the claim is true for $n = 1$.

Now assume that the claim is true for some positive integer n and consider a tournament with 2^{n+1} players. One can consider the top half of the draw as a tournament with 2^n players that produces the finalist with the higher rank and the bottom half of the draw as a tournament with 2^n players that produces the finalist with the lower rank. Let W_+ be the random variable corresponding to the rank of the finalist from the top half of the draw and let W_- be the random variable corresponding to the rank of the finalist from the bottom half of the draw. Then we have $E[W_-] = E[W_+] + 2^n$ and $E[W_+] = 2^n - 2^n p + p$ by the induction hypothesis. Therefore,

$$\begin{aligned} E[W] &= p \times E[W_+] + (1-p) \times E[W_-] \\ &= p \times (2^n - 2^n p + p) + (1-p) \times (2^n - 2^n p + p + 2^n) \\ &= 2^n - 2^n p + p + 2^n - 2^n p \\ &= 2^{n+1} - 2^{n+1} p + p. \end{aligned}$$

Thus, the claim is true for a tournament with 2^{n+1} players. By induction, we have proven the claim for a tournament with 2^n players for every positive integer n.

Finally, the statement of the problem uses $n = 8$ and $p = \frac{3}{5}$, in which case the expected value of the rank of the winner is

$$2^8 - 2^8 \times \frac{3}{5} + \frac{3}{5} = 103.$$

Solution 2

More generally, suppose that there are 2^n players, so that the tournament lasts for n rounds. Furthermore, suppose that the person whose rank is higher wins the match with probability p. Let W be the random variable corresponding to the rank of the winner. We will prove that

$$E[W] = 2^n - 2^n p + p.$$

Consider the player whose rank is k and the probability with which they win the tournament. Express the integer $k-1$ in binary, but pad it out with preceding 0s to create a binary string of length n. Observe that in round r, this player plays someone with higher rank if the rth last binary digit of the string is 1; similarly, they play someone with lower rank if the rth last binary digit of the string is 0.

So the probability that the player whose rank is k wins the tournament is equal to $p^{n-b(k-1)}(1-p)^{b(k-1)}$, where $b(k-1)$ represents the number of times that a 1 appears in the binary representation of $k-1$. The quantity that we would like to calculate is

$$\begin{aligned}
E[W] &= \sum_{k=1}^{2^n} k \times p^{n-b(k-1)}(1-p)^{b(k-1)} \\
&= \sum_{k=0}^{2^n-1} (k+1) \times p^{n-b(k)}(1-p)^{b(k)} \\
&= \sum_{b=0}^{n} \sum_{k:b(k)=b} (k+1)p^{n-b}(1-p)^b \\
&= \sum_{b=0}^{n} p^{n-b}(1-p)^b \sum_{k:b(k)=b} (k+1).
\end{aligned}$$

The inner summation is simply the sum of all the non-negative integers whose binary representation contains at most n digits, precisely k of which are 1. There are exactly $\binom{n}{k}$ such integers and in total, there are $k\binom{n}{k}$ times that 1 appears as a binary digit amongst these integers. So by symmetry, 1 appears as the mth binary digit from the right $\frac{k}{n}\binom{n}{k}$ times for each $m = 1, 2, \ldots, n$. It then follows that we have

$$\sum_{k:b(k)=b} b = \frac{b}{n}\binom{n}{b} \times (2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}) = \frac{b}{n}\binom{n}{b} \times (2^n - 1).$$

Substituting this into the expression above, we obtain that the expected value of the winner's rank is

$$\begin{aligned}
&\sum_{b=0}^{n} p^{n-b}(1-p)^b \frac{b}{n}\binom{n}{b} \times (2^n - 1) + 1 \\
&= \frac{2^n-1}{n} \sum_{b=0}^{n} p^{n-b}(1-p)^b b\binom{n}{b} + 1 \\
&= \frac{2^n-1}{n} \left[x\frac{\partial}{\partial x} \sum_{b=0}^{n} y^{n-b}x^b \binom{n}{b} \right]_{x=1-p, y=p} + 1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2^n - 1}{n} \left[x \frac{\partial}{\partial x} (x+y)^n \right]_{x=1-p, y=p} + 1 \\
&= \frac{2^n - 1}{n} \left[nx(x+y)^{n-1} \right]_{x=1-p, y=p} + 1 \\
&= (2^n - 1)(1-p) + 1 \\
&= 2^n - 2^n p + p.
\end{aligned}
$$

Finally, the statement of the problem uses $n = 8$ and $p = \frac{3}{5}$, in which case the expected value of the rank of the winner is

$$
2^8 - 2^8 \times \frac{3}{5} + \frac{3}{5} = 103.
$$

Problem B3

Let n be a positive integer. Let A, B and C be three n-dimensional subspaces of $\mathbb{R}^{2n}$ with the property that $A \cap B = B \cap C = C \cap A = \{\mathbf{0}\}$. Prove that there exist bases $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ of A, $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of B and $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ of C with the property that for each $i \in \{1, 2, \ldots, n\}$, the vectors $\mathbf{a}_i$, $\mathbf{b}_i$ and $\mathbf{c}_i$ are linearly dependent.

Solution 1

Let $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ be a basis of C. Since $A \cap B = \{\mathbf{0}\}$ and $\dim A + \dim B = \dim \mathbb{R}^{2n}$, we have $A + B = \mathbb{R}^{2n}$. Therefore for each $i \in \{1, 2, \ldots, n\}$, there exists $\mathbf{a}_i \in A$ and $\mathbf{b}_i \in B$ with $\mathbf{a}_i + \mathbf{b}_i = \mathbf{c_i}$. It suffices to show that $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is a basis of A and $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis of B.

Since A is n-dimensional, to show $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is a basis of A it suffices to show that $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is a linearly independent set. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ are such that $\sum_{i=1}^n \lambda_i \mathbf{a}_i = \mathbf{0}$. Then

$$\sum_{i=1}^n \lambda_i \mathbf{b}_i = \sum_{i=1}^n \lambda_i \mathbf{c}_i.$$

This vector lies in both B and C. Since $B \cap C = \{\mathbf{0}\}$, we obtain $\sum_{i=1}^n \lambda_i \mathbf{c}_i = \mathbf{0}$. Since $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ is a basis of C, it is a linearly independent set, so $\lambda_i = 0$ for all i, as required.

This shows that $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is a basis of A. Similarly $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis of B. The equation $\mathbf{a}_i + \mathbf{b}_i = \mathbf{c_i}$ shows that $\mathbf{a}_i$, $\mathbf{b}_i$, and $\mathbf{c}_i$ are linearly independent for all i. This completes the proof.

Solution 2

Let $\varphi : A \oplus B \oplus C \to \mathbb{R}^{2n}$ be the linear transformation $\varphi(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} + \mathbf{b} + \mathbf{c}$. Then $\dim(\ker \varphi) \geq \dim(A \oplus B \oplus C) - \dim(\mathbb{R}^{2n}) = n$. Let $\{(\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i)\}_{1 \leq i \leq n}$ be n linearly independent vectors in $\ker \varphi$. Note that $\mathbf{a}_i + \mathbf{b}_i + \mathbf{c}_i = \mathbf{0}$, so the vectors $\mathbf{a}_i$, $\mathbf{b}_i$, and $\mathbf{c}_i$ are linearly dependent for all i.

We will now show that the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent. Suppose that there exist scalars λ_i such that

$$\lambda_1 \mathbf{a}_1 + \cdots + \lambda_n \mathbf{a}_n = \mathbf{0}.$$

Then

$$(\lambda_1 \mathbf{b}_1 + \cdots + \lambda_n \mathbf{b}_n) = -(\lambda_1 \mathbf{c}_1 + \cdots + \lambda_n \mathbf{c}_n).$$

Since $B \cap C = \{\mathbf{0}\}$, this implies that $\lambda_1 \mathbf{b}_1 + \cdots + \lambda_n \mathbf{b}_n = \mathbf{0}$ and $\lambda_1 \mathbf{c}_1 + \cdots + \lambda_n \mathbf{c}_n = \mathbf{0}$. Therefore

$$\lambda_1(\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}_1) + \lambda_2(\mathbf{a}_2, \mathbf{b}_2, \mathbf{c}_2) + \cdots + \lambda_n(\mathbf{a}_n, \mathbf{b}_n, \mathbf{c}_n) = 0.$$

Since the vectors $(\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i)$ are linearly independent, we deduce $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, which proves that $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a linearly independent set, and hence is a basis of A since A is n-dimensional.

Similarly $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis of B and $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ is a basis of C, completing the proof.

Problem B4

(a) Let n be a positive integer that is not a perfect square. Find all pairs (a, b) of positive integers for which there exists a positive real number r, such that

$$r^a + \sqrt{n} \quad \text{and} \quad r^b + \sqrt{n}$$

are both rational numbers.

(b) Let n be a positive integer that is not a perfect square. Find all pairs (a, b) of positive integers for which there exists a real number r, such that

$$r^a + \sqrt{n} \quad \text{and} \quad r^b + \sqrt{n}$$

are both rational numbers.

Solution to part (a)

We will show that for every value of n, the only solution is $a = b$. When $a = b$, there is indeed a solution, by taking $r = (n - \sqrt{n})^{\frac{1}{a}}$.

Let $p = r^a + \sqrt{n}$ and $q = r^b + \sqrt{n}$. Then

$$(p - \sqrt{n})^b = (q - \sqrt{n})^a. \tag{2}$$

Since $\sqrt{n}$ is irrational, by considering conjugates we also get that

$$(p + \sqrt{n})^b = (q + \sqrt{n})^a. \tag{3}$$

If $p = q$ then $r^a = r^b$ and hence either $a = b$ or $r = 1$, and $r = 1$ is not possible since $\sqrt{n}$ is irrational.

Now suppose without loss of generality that $p < q$. Then from (3), $b > a$. Consider the function

$$f(x) = b \log(p + x) - a \log(q + x).$$

Its derivative is

$$f'(x) = \frac{b}{p + x} - \frac{a}{q + x}$$

and for $x > -p$, we have $f'(x) > 0$ since $p < q$ and $b > a$.

Note that $r^a = p - \sqrt{n}$ so $-\sqrt{n} > -p$. Therefore f is an increasing function on the interval $[-\sqrt{n}, \sqrt{n}]$. But equations (2) and (3) are equivalent to $f(-\sqrt{n}) = 0 = f(\sqrt{n})$, a contradiction.

Hence $a = b$ is the only solution.

Known partial results for part (b)

We will show that for every n there is the additional solution $\{a, b\} = \{1, 2\}$, and that any other solution must have $a, b > 1$ with a and b of opposite parity. The solution with $(a, b) = (1, 2)$ is

$$r = \frac{1}{2} - \sqrt{n}$$

which is easily checked to be a solution.

If a and b are both even, then replacing r by $|r|$, we can assume without loss of generality that r is positive and there is nothing new to be done.

Now suppose a and b are both odd. Without loss of generality assume $a \leq b$. We have

$$(p - \sqrt{n})^{b/a} = q - \sqrt{n}.$$

As before, we have $(p + \sqrt{n})^b = (q + \sqrt{n})^a$ and taking the a-th root gives

$$(p + \sqrt{n})^{b/a} = q + \sqrt{n}.$$

If $p > \sqrt{n}$ then there are no solutions by the argument in part (a). If $p < \sqrt{n}$ then replacing p by $-p$ also yields no solution by the argument in part (a). So we can suppose that $-\sqrt{n} < p < \sqrt{n}$.

Let $Y = \sqrt{n} - p$ and $Z = \sqrt{n} + p$. Then Y and Z are positive. We obtain $Y + Z = 2\sqrt{n}$ and $Y^{b/a} + Z^{b/a} = 2\sqrt{n}$. By the power mean inequality (or since $x \mapsto x^{b/a}$ is convex),

$$\left(\frac{Y^{b/a} + Z^{b/a}}{2}\right)^{a/b} \geq \frac{Y + Z}{2},$$

which yields

$$n^{a/b} \geq n.$$

Since n is not a perfect square, $n > 1$. Therefore $a/b \geq 1$ but since we assumed $a \leq b$, we get $a = b$ and so there are no additional solutions.

We now give a complete solution to the $a = 1$ case. We will show that the only solutions when $a = 1$ are $b = 1$ and $b = 2$, for which solutions are known and discussed above. The case where b is odd has been dealt with above, so assume $b = 2m$ with $m \geq 2$.

The equation we have to solve is

$$(p + \sqrt{n})^{2m} = q + \sqrt{n}.$$

Since 1 and $\sqrt{n}$ are linearly independent over $\mathbb{Q}$, we get

$$\sum_{k=1}^{m} \binom{2m}{2k-1} p^{2k-1} n^{m-k} = 1. \tag{4}$$

Let v_2 denote the 2-adic valuation, this is the function $v_2 : \mathbb{Q}^\times \to \mathbb{Z}$ defined by $v_2(2^a \frac{b}{c}) = a$, where $a, b, c \in \mathbb{Z}$ with b, c odd. For all $k = 1, 2, \ldots, m$,

$$\binom{2m}{2k-1} = \frac{2m}{2m - 2k + 1} \binom{2m-1}{2k-1}.$$

Since $2m - 2k + 1$ is odd, we get

$$v_2\left(\tbinom{2m}{2k-1}\right) = v_2(2m) + v_2\left(\tbinom{2m-1}{2k-1}\right) > 0 \tag{5}$$

If $v_2(p) \geq 0$ then (5) implies the left hand side of (4) has positive 2-adic valuation, while the right hand side of (4) has zero 2-adic valuation, a contradiction.

If $v_2(p) < 0$, then (5) implies that

$$v_2(2mp^{2m-1}) < v_2\left(\tbinom{2m}{2k-1}p^{2k-1}n^{m-k}\right)$$

for $k = 1, 2, \ldots, m-1$. Therefore the left hand side of (4) has 2-adic valuation equal to

$$v_2(2m) + (2m-1)v_2(p).$$

As $v_2(p) \leq -1$ and $m \geq 2$, this quantity is negative. The right hand side of (4) has zero 2-adic valuation so this is also a contradiction.

Therefore there are no solutions with $a = 1$ and $b > 2$.

Problem C1

There are 2023 cups numbered from 1 through 2023. Red, green, and blue balls are placed in the cups according to the following rules.

- If cups m and n both contain a red ball, then $m - n$ is a multiple of 2.
- If cups m and n both contain a green ball, then $m - n$ is a multiple of 3.
- If cups m and n both contain a blue ball, then $m - n$ is a multiple of 5.

What is the smallest possible number of empty cups?

Solution

We may assume that there exists at least one ball of each other colour since otherwise there will be more empty cups.

The conditions imply that there exist integers a, b, and c such that:

- Cup n contains a red ball if and only if $n \equiv a \pmod 2$.
- Cup n contains a green ball if and only if $n \equiv b \pmod 3$.
- Cup n contains a blue ball if and only if $n \equiv c \pmod 5$.

By the Chinese remainder theorem, we conclude that there must exist an integer x such that Cup n is empty if and only if $\gcd(n - x, 30) = 1$.

Since the number of positive integers below 30 that are coprime with 30 is $\varphi(30) = 8$, there must be exactly 8 cups in any sequence of 30 cups that satisfy this condition.

Note that $2023 = 67 \times 30 + 13$. The 8 residue classes coprime to 30 are

$$1, 7, 11, 13, 17, 19, 23, 29.$$

There is a sequence of 13 consecutive residue classes (starting at 24 and finishing at 6) that contains exactly two members of this set. It can be easily checked that there is no sequence of 13 consecutive residue classes containing fewer than two members of this set.

Therefore the smallest possible number of empty cups amongst 13 consecutive cups is 2, and hence the smallest possible number of empty cups is

$$67 \times 8 + 2 = 538.$$

Problem C2

For an integer $n \geq 2$, consider the line segment connecting the point $(0, k)$ to the point $(n-k, 0)$ for $k = 0, 1, 2, \ldots, n$. The union of these $n+1$ line segments divides the plane into one unbounded region and a number of bounded regions, each of which is a triangle or a quadrilateral. Each of these bounded regions can be coloured blue or red in a unique way such that regions sharing an edge have different colours and the region with vertex $(0, 0)$ is coloured blue.

Determine all values of n for which the total area that is coloured blue is equal to the total area that is coloured red.

Solution

For $k = 0, 1, 2, \ldots, n$, let ℓ_k denote the line connecting the point $(0, k)$ to the point $(n-k, 0)$. For $0 \leq i \leq j \leq n$, define the point

$$P_{ij} = \left(\frac{(n-i)(n-j)}{n}, \frac{ij}{n} \right).$$

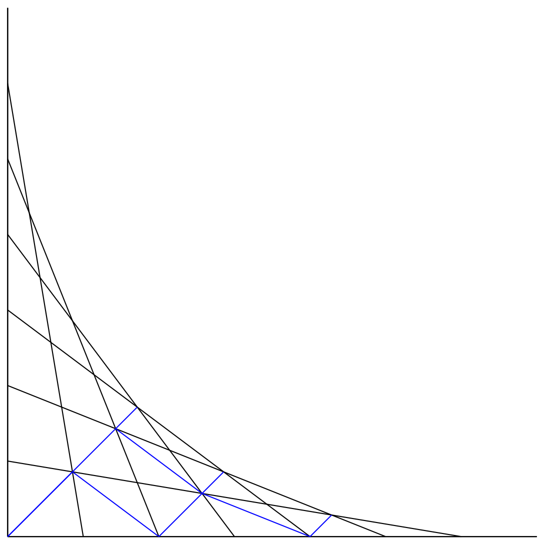
One can check that for $i < j$, the point P_{ij} is the point where the lines ℓ_i and ℓ_j intersect. Furthermore, observe that for $j = 0, 1, 2, \ldots, n$, the $n+1$ points

$$P_{0j}, P_{1j}, P_{2j}, \ldots, P_{j-1,j}, P_{jj}, P_{j,j+1}, \ldots, P_{jn}$$

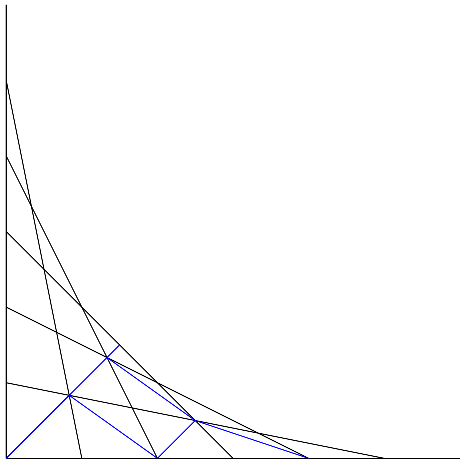
are equally spaced along ℓ_j in order, with P_{0j} and P_{jn} forming the endpoints.

We will prove that the area that is coloured blue is equal to the area that is coloured red if and only if n is odd.

Let n be odd and consider the part of the diagram below the line $y = x$. By drawing the blue segments shown below in the case of $n = 7$, we split this part of the diagram into triangles, each of which pairs with an adjacent triangle with the same area but different colour. The fact that the areas are equal follows from the fact above concerning the equidistant spacing of the points along each line. It follows that the area that is coloured blue is equal to the area that is coloured red.



Now let n be even and consider the part of the diagram below the line $y = x$. By drawing the blue segments shown below in the case of $n = 6$, we split this part of the diagram into triangles, most of which which pair with an adjacent triangle with the same area but different colour. The fact that the areas are equal follows from the fact above concerning the equidistant spacing of the points along each line. However, in this case, the "edge" triangles are blue and are twice the area of the triangle that they pair with. It follows that the area that is coloured blue is greater than the area that is coloured red.



Problem C3

Determine the maximum real number C such that

$$\sum_{i=1}^{n} \frac{x_i}{x_{i-1}} \geq n + C,$$

for all positive integers n and all sequences of positive real numbers $x_0, x_1, \ldots, x_n$ such that $x_0 = 1$ and $x_n = 2$.

Solution 1

We will show that $C = \log(2)$ (where log refers to the natural logarithm).

By the AM-GM inequality we can compute:

$$\frac{\frac{x_1}{x_0} + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \cdots + \frac{x_n}{x_{n-1}}}{n} \geq \sqrt[n]{\frac{x_1}{x_0} \times \frac{x_2}{x_1} \times \frac{x_3}{x_2} \times \cdots \times \frac{x_n}{x_{n-1}}} = \sqrt[n]{2},$$

with equality if and only if there is some constant $r = \frac{x_i}{x_{i-1}}$ for all i. So equality can be achieved by $x_i = 2^{i/n}$ ($r = \sqrt[n]{2}$) for all $i = 0, 1, 2, 3, \ldots, n$. Define

$$S_n = n\left(\sqrt[n]{2} - 1\right)$$

and thus it suffices to:

- show S_n is a decreasing sequence, and
- determine $C = \lim_{n\to\infty} S_n$.

Letting $t = \frac{1}{n}$, we see

$$\begin{aligned} n(\sqrt[n]{2} - 1) &= \tfrac{1}{t}(2^t - 1) \\ &= \tfrac{1}{t}(e^{t\log 2} - 1) \\ &= \log 2 + \frac{(\log 2)^2 t}{2} + \frac{(\log 2)^3 t^2}{6} + \frac{(\log 2)^4 t^3}{24} + \cdots \end{aligned}$$

using the power series formula for the exponential function.

As this is a sum of decreasing functions of t, it is clear that we have a decreasing function of t and its limit is $\log 2$. This completes the proof, and shows that $C = \log 2$.

Solution 2

Let $f(x)=\frac{1}{x}$. First let us consider the case when $1=x_0<x_1<x_2<\cdots<x_n=2$. Since $f(x)$ is decreasing on the interval $[1,2]$, the upper Riemann sum for $f(x)$ for this partition of $[1,2]$ is

$$\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)f\left(x_{i-1}\right).$$

This upper sum is greater than the integral so we get:

$$\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)\frac{1}{x_{i-1}}\geq\int_1^2\frac{1}{x}dx=\log(2).$$

When we no longer have the increasing property $1=x_0<x_1<x_2<\cdots<x_n=2$, we can still consider the sum

$$\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)f\left(x_{i-1}\right).$$

Individual terms in this sum may now be negative, we can think of it as a signed sum of areas of the rectangles with vertices $(x_{i-1},0),(x_i,0),(x_i,f(x_i)),(x_{i-1},f(x_{i-1}))$.

Any point in the plane appears in at least as many rectangles with a positive sign than with a negative sign. Each point below the curve $y=f(x)$ (and with $1<x<2$, $y\geq 0$) is in strictly more rectangles with a positive sign than with a netative sign. So by conidering areas we obtain the same inequality

$$\sum_{i=1}^{n}\left(x_i-x_{i-1}\right)\frac{1}{x_{i-1}}\geq\int_1^2\frac{1}{x}dx=\log(2).$$

Rearranging gives us

$$\sum_{i=1}^{n}\left(\frac{x_i}{x_{i-1}}-1\right)\geq\log(2).$$

$$\sum_{i=1}^{n}\frac{x_i}{x_{i-1}}\geq n+\log(2).$$

So $C\geq\log(2)$.

Now for any given $\epsilon>0$ we need to construct a sequence such that

$$\sum_{i=1}^{n}\frac{x_i}{x_{i-1}}<n+\log(2)+\epsilon.$$

Since the Riemann integral is the limit of its upper sums, we can find increasing sequences $1=x_0<x_1<x_2<\ldots<x_n=2$ that satisfy the above inequality.

Problem C4

Let k be a positive integer. Keira and Roland play a game of *reverse chess*. Initially, Roland chooses a positive integer $n > \frac{k}{2023}$. Keira places k kings on k distinct squares of a $2023 \times n$ chess board. Then Roland places a rook on an unoccupied square of the board. Both players then take turns moving any number (possibly zero) of their pieces, with Keira starting first. Each king cannot move to a square occupied by another king, but it can capture the rook. Furthermore, if Keira chooses to move more than one king in her turn, she moves them one at a time. Roland's rook is not permitted to capture any king, nor may it pass through a square occupied by a king.

For which k can Keira guarantee to capture Roland's rook, regardless of Roland's moves or choice of n?

(A king can move exactly one square in any horizontal, vertical or diagonal direction. A rook can move any number of squares in a horizontal or vertical direction. One piece (rook or king) captures another by moving to the square it occupies.)

Solution

We will show the answer is $k \geq 1012$.

We first show that Roland can avoid capture if $k \leq 1011$. Let n be the number of rows. Choose $n > 2027 \times 1012$, so that there always exists a row which is at least 1014 rows away from any rows occupied by a king. Call such rows *safe*. We will show that Roland's rook can avoid capture by moving between safe rows.

Initially, Roland places his rook in a safe row. After Keira's first turn, Roland moves the rook to the leftmost column. In each subsequent turn, Roland checks whether there is a king in the same column as his rook.

- If there is no king in the same column, then Roland can move the rook vertically to a safe row.
- If there is a king in the same column, then Roland will move the rook two squares to the right.

The second case can occur at most 1011 times, since a king can only move by at most one column per turn, so a different king is needed to block the rook's column each turn. After at most 1012 turns (including the first turn where Roland moved the rook to the leftmost column), Roland can move his rook vertically to a new safe row. Note that Keira will have taken at most 1013 turns during this, which is not enough to capture the rook which started on a safe row.

By repeating this method, Roland's rook can move between safe rows and avoid capture indefinitely if $k \leq 1011$.

We now show that $k = 1012$ (and hence $k \geq 1012$) kings are enough to capture the rook.

Keira moves the kings until there is a king in each of the topmost squares of columns $1, 3, \ldots, 2023$. Label these kings with $K_1, \ldots, K_{1012}$, from left to right. For each i, assign

K_i to guard columns $2i-1$ and $2i$ (with K_{1012} only guarding the rightmost column). After achieving this, let us assume that the rook is not in the topmost column on Keira's move, otherwise the kings can immediately capture it.

The kings move down by one row every turn, so that each king moves to one of the two squares in its assigned double-column in the row below. If the rook is in the double-column guarded by K_i, then K_i will choose its move so that it is in the same column as the rook. As a result, the rook must always be below the row occupied by the kings. The game will eventually reach a state where, after its move, the rook is in the row directly below the kings. Keira can then capture the rook on her next move.