SIMON MARAIS mathematics competition

2024 PAPER B: INSTRUCTIONS

Time allowed: 3 hours, with no additional reading time.

Each problem B1–B3 is worth 7 points. Points for problem B4 are as indicated.

Partial credit may be awarded for an incomplete solution or progress towards a solution.

Instructions for all contestants

- This is a **closed-book** examination. No notes, books, calculators, electronic devices or other aids are allowed to assist in answering the questions. Tablets may be used solely for writing worked solutions, with internet access switched off.
- For participants sitting the exam off-site, an electronic device such as a PC, laptop, phone or tablet may be used during the competition for accessing the papers, undergoing invigilation, writing and submitting solutions and (for pairs entrants) communicating with the other member of the pair.
- Write your solutions in English, using a black or blue pen on white or light-coloured paper, or on a tablet.
- **In the top left corner of every page**, write the competition ID number you have been assigned. **Do not** write your name, or anything else that could identify you or your university. You may write your ID number before the start of the session.
- **In the top right corner of every page**, write the problem number it relates to, and the page number **within that problem** — for example, "B3 P2". Each page must relate to only one problem.
- If a particular problem is **not attempted**, a page marked with your competition ID number and the problem number as per the instructions above should be submitted.
- Students are strongly encouraged to submit all rough work pages as they may lead to partial credit. Students are also allowed to submit more than one attempted solution per problem. All pages for a single problem (including rough work and multiple solution attempts) should be numbered in one sequence.
- After the completion of the session all participants should scan their work and convert the scan into a single PDF file. This PDF file, labelled by your competition ID number and the paper (as in **S1234567B** (for singles) or **P3141593B** (for pairs)), should be e-mailed to your local coordinator within **30 minutes** of the completion of the session.

Special instructions for pairs

- A pair should make only one submission for each problem. Pages should be labelled with the competition ID number assigned to the pair as well as the page numbering indicated above.
- Make sure that your discussions are not overheard by other contestants.

SIMON MARAIS mathematics competition

2024 PAPER B: PROBLEMS

B1. Alice has six boxes labelled 1 through 6. She secretly chooses exactly two of the boxes and places a coin inside each. Bob is trying to guess which two boxes contain the coins. Each time Bob guesses, he does so by tapping exactly two of the boxes. Alice then responds by telling him the total number of coins inside the two boxes that he tapped. Bob successfully finds the two coins when Alice responds with the number 2.

What is the smallest positive integer n such that Bob can always find the two coins in at most n guesses?

B2. Determine all continuous functions $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ that satisfy

$$f(x) = (x + 1)\, f(x^2),$$

for all $x \in \mathbb{R} \setminus \{-1, 1\}$.

B3. Let $\mathcal{L}$ be the set of all lines in the plane and let $\mathcal{P}$ be the set of all points in the plane.

Determine whether there exists a function $g : \mathcal{L} \to \mathcal{P}$ such that for any two distinct non-parallel lines $\ell_1, \ell_2 \in \mathcal{L}$, we have

(a) $g(\ell_1) \neq g(\ell_2)$, and

(b) if ℓ_3 is the line passing through $g(\ell_1)$ and $g(\ell_2)$, then $g(\ell_3)$ is the intersection of ℓ_1 and ℓ_2.

B4. *The following problem is open in the sense that the answer to part (b) is not currently known. A solution to part (a) will be awarded 7 points. Up to 7 additional points may be awarded for progress on part (b).*

Let n be an odd positive integer and let

$$f_n(x, y, z) = x^n + y^n + z^n + (x + y + z)^n.$$

(a) Prove that there exist infinitely many values of n such that

$$f_n(x, y, z) \equiv (x + y)(y + z)(z + x)\, g(x, y, z)\, h(x, y, z) \pmod 2,$$

for some integer polynomials $g(x, y, z)$ and $h(x, y, z)$, neither of which is constant modulo 2.

(b) Determine all values of n such that

$$f_n(x, y, z) \equiv (x + y)(y + z)(z + x)\, g(x, y, z)\, h(x, y, z) \pmod 2,$$

for some integer polynomials $g(x, y, z)$ and $h(x, y, z)$, neither of which is constant modulo 2.

(Two integer polynomials are *congruent modulo 2* if every coefficient of their difference is even. A polynomial is *constant modulo 2* if it is congruent to a constant polynomial modulo 2.)

