SIMON MARAIS mathematics competition

2024 SOLUTIONS

Problem A1 (proposed by Norman Do)

Let a, b, c be real numbers greater than 1 satisfying

$$\lfloor a \rfloor b = \lfloor b \rfloor c = \lfloor c \rfloor a.$$

Prove that $a = b = c$.

(Here, $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.)

Solution 1

The equations are cyclically symmetric so suppose without loss of generality that $a \le b$ and $a \le c$. Throughout we will continually use the fact that $\lfloor a \rfloor$, $\lfloor b \rfloor$ and $\lfloor c \rfloor$ are positive without comment.

In the equation $\lfloor a \rfloor b = \lfloor b \rfloor c$ we have $\lfloor a \rfloor \le \lfloor b \rfloor$ and hence $b \ge c$.

In the equation $\lfloor b \rfloor c = \lfloor c \rfloor a$ we have $\lfloor b \rfloor \ge \lfloor c \rfloor$ and hence $c \le a$. Combined with the inequality that $a \le c$, we've shown that $a = c$.

Substitute $c = a$ into the equation $\lfloor c \rfloor a = \lfloor a \rfloor b$ and we deduce $b = a$. Hence we've shown $a = b = c$ as required.

Solution 2

We have

$$\frac{\lfloor c \rfloor}{b} = \frac{\lfloor a \rfloor}{a} \le 1$$

which implies that $\lfloor c \rfloor \le b$. By cyclic symmetry we similarly obtain $\lfloor b \rfloor \le a$ and $\lfloor a \rfloor \le c$.

Putting these inequalities together yields the chain of inequalities

$$b \ge \lfloor c \rfloor > c - 1 \ge \lfloor a \rfloor - 1 > a - 2 \ge \lfloor b \rfloor - 2 > b - 3.$$

Therefore the three distinct integers $\lfloor c \rfloor > \lfloor a \rfloor - 1 > \lfloor b \rfloor - 2$ all lie in the interval $(b - 3, 3]$ of length three. The only way for this to be possible is that these three integers are consecutive. Hence $\lfloor a \rfloor = \lfloor b \rfloor = \lfloor c \rfloor$.

As $a, b, c \ge 1$, we can cancel out the common nonzero factor $\lfloor a \rfloor = \lfloor b \rfloor = \lfloor c \rfloor$ from the given equations to conclude that $a = b = c$.

Problem A2 (proposed by Michael Albert and Christopher Tuffley)

A positive integer n is *tripairable* if it is possible to partition the set $\{1, 2, \ldots, n\}$ into disjoint pairs such that the sum of the two elements in each pair is a power of 3. For example, 6 is tripairable because $\{1, 2, 3, 4, 5, 6\} = \{1, 2\} \cup \{3, 6\} \cup \{4, 5\}$ and

$$1 + 2 = 3^1, \quad 3 + 6 = 3^2 \quad \text{and} \quad 4 + 5 = 3^2$$

are all powers of 3.

How many positive integers less than or equal to 2024 are tripairable?

Solution 1

We will say that a partition of $[n] = \{1, 2, \ldots, n\}$ into pairs is a *tripairing* if it satisfies the condition given in the problem. Clearly for n to be tripairable, it must be even. It will be convenient to extend the definition to $n = 0$, which we will regard as tripairable by considering the empty partition of the empty set as a tripairing.

We will show that n is tripairable if and only if the ternary expansion of n contains only 0s and 2s. More precisely, we will prove by induction on k that for all $k \geq 1$, if $n < 3^k$ then n is tripairable if and only if the ternary expansion of n contains only 0s and 2s.

For the base case $k = 0$, note that the only non-negative integer less than 3^0 is 0. The number zero is tripairable and only has 0s and 2s in its ternary expansion, completing the base case.

Now suppose that the inductive hypothesis is true for some $k \geq 1$, and consider n such that $3^k \leq n < 3^{k+1}$. If $3^k \leq j \leq n$ then j must be paired with $3^{k+1} - j$ in any tripairing of $[n]$, in order for the sum condition to be satisfied. This completely determines the partner of j in any tripairing of $[n]$ for $3^{k+1} - n \leq j \leq n$. Since n is even the numbers j and $3^{k+1} - j$ are distinct. It follows that the number of tripairings of $[n]$ is equal to the number of tripairings of $[m]$, where $m = 3^{k+1} - n - 1$.

Now observe in particular that 3^k must be paired with $3^{k+1} - 3^k = 2 \cdot 3^k$. This implies $2 \cdot 3^k \leq n$. But then $m = 3^{k+1} - n - 1 \leq 3^{k+1} - 2 \cdot 3^k - 1 = 3^k - 1$, so the inductive hypothesis applies to m. Thus we may write

$$m = 2 \sum_{i \in I} 3^i,$$

for some subset I of $\{0\} \cup [k-1]$. Then

$$\begin{aligned} n &= 3^{k+1} - m - 1 \\ &= 2 \sum_{i=0}^{k} 3^i - 2 \sum_{i \in I} 3^i \\ &= 2 \sum_{i \in J} 3^i, \end{aligned}$$

where $J = (\{0\} \cup [k]) - I$ is the complement of I in $\{0\} \cup [k]$. This shows that the ternary expansion of n also consists only of 0s and 2s.

Conversely, suppose that $3^k \le n < 3^{k+1}$ and the ternary expansion of n consists of 0 and 2s only. Then we may write

$$n = 2 \cdot 3^k + 2\sum_{i \in I} 3^i,$$

for some subset I of $\{0\} \cup [k-1]$. Then

$$\begin{aligned} m &= 3^{k+1} - n - 1 \\ &= 2\sum_{i=0}^{k} 3^i - 2 \cdot 3^k - 2\sum_{i \in I} 3^i \\ &= 2\sum_{i \in J} 3^i, \end{aligned}$$

where $J = (\{0\} \cup [k-1]) - I$ is the complement of I in $\{0\} \cup [k-1]$. By the inductive hypothesis m is tripairable, so n is too. This completes the proof of the inductive step.

Now we count the number of tripairable nonnegative integers n satisfying $n \le 2024$. We have

$$2024 = 2(729 + 243 + 27 + 9 + 3 + 1),$$

so the ternary expansion of 2024 is 2202222_3. Thus if $0 \le n \le 2024$ is tripairable, its ternary expansion must have one of the following forms, where $a_i \in \{0, 2\}$ for all i:

(a) $a_5a_4a_3a_2a_1a_0$

(b) $20a_4a_3a_2a_1a_0$

(c) $220a_3a_2a_1a_0$.

There are 2^6 numbers of the first form, 2^5 of the second, and 2^4 of the third, so accounting for $n = 0$ there are

$$2^6 + 2^5 + 2^4 - 1 = 111$$

tripairable positive integers n with $n \le 2024$.

Solution 2

We first note that if n is tripairable then n is even.

Lemma 1. *If there exists an integer a such that $3^a \le n < 2 \cdot 3^a$ then n is not tripairable.*

Proof. Suppose n is tripairable and let b be the element paired with 3^a in our tripairing of n. Then $3^a + b = 3^c$ for some integer c. As $b > 0$, we have $c > a$. But as $b \le n < 2 \cdot 3^a$ we have $c < a + 1$. This is a contradiction since c is an integer. □

Lemma 2. *Suppose there exists an integer a such that $2 \cdot 3^a \le n < 3^{a+1}$. Then n is tripairable if and only if $3^{a+1} - n - 1$ is tripairable, or is zero.*

Proof. Let b be an integer with $3^a \le b \le n$. Consider a tripairing of n. Since $3^a - b \le 0$ and $3^{a+2} - b > n$, the number b must be paired with $3^{a+1} - b$.

This implies that every number b from $3^{a+1} - n$ to n inclusive is paired with $3^{a+1} - b$. What remains to be paired are the positive integers up to $3^{a+1} - n - 1$, implying the lemma. □

Corollary 1. *Let a be a positive integer. Then there are $2^a - 1$ tripairable numbers less than or equal to 3^a.*

Proof. We proceed by induction on a. The base case is $a = 1$, where there is one tripairable number, namely 2. For the inductive step, suppose $a \geq 2$ and we know the result for $a-1$. The two lemmas above imply that the number of tripairable numbers between 3^{a-1} and 3^a is equal to 1 more than the number of tripairable numbers less than 3^{a-1}. By the inductive hypothesis, the number of tripairable numbers less than or equal to 3^a is therefore

$$(2^{a-1} - 1) + ((2^{a-1} - 1) + 1) = 2^a - 1.$$

as required. ∎

We now count the number of tripairable numbers less than or equal to 2024. By the Corollary, there are 63 tripairable numbers less than or equal to 729. By Lemma 1, the number of tripairable numbers in $[730, 2024]$ is equal to the number of tripairable numbers in $[1458, 2024]$, which by Lemma 2 is equal to the number of tripairable numbers in $[162, 729]$. By Lemma 1, this is equal to the number of tripairable numbers in $[82, 729]$, which by the Corollary, is equal to $63 - 15 = 48$.

Thus the total number of tripairable positive integers that are less than or equal to 2024 is $63 + 48 = 111$.

Problem A3 (proposed by Johan Meyer)

Let W be a fixed positive integer. Let S be the set of all pairs (a,b) of positive integers such that $a \neq b$. For each $(a,b) \in S$, let $m(a,b)$ be the largest integer satisfying $m(a,b) \leq \dfrac{1+na}{1+nb}$ for all integers $n \geq 1$.

(a) For each $(a,b) \in S$, prove that there exists a positive integer k such that

$$m(a,b) \leq \frac{1+na}{W+nb},$$

for all $n \geq k$.

(b) For each $(a,b) \in S$, let $k(a,b)$ be the smallest value of k that satisfies the condition of part (a). Determine $\max\{k(a,b) \mid (a,b) \in S\}$ or prove that it does not exist.

Solution 1

We have

$$\frac{1+na}{1+nb} = \frac{a}{b} + \frac{1-\frac{a}{b}}{1+nb}.$$

If $a < b$ then this is a decreasing sequence with limit $\frac{a}{b}$ which is strictly smaller than 1. Hence $m(a,b) = 0$ in this case. Furthermore $0 \leq \frac{1+na}{1+nb}$ for all positive integers n, and hence we can take $k = 1$ in the inequality of part (a). So $k(a,b) = 1$ when $a < b$.

We now assume $a > b$. In this case the sequence $\frac{1+na}{1+nb}$ is an increasing sequence. So it takes its minimum value when $n = 1$ and hence

$$m(a,b) = \left\lfloor \frac{1+a}{1+b} \right\rfloor.$$

We will show that the maximum value in part (b) is $2W-1$, proving part (a) in the process. The case $a < b$ has been taken care of since $2W - 1 \geq 1$. Now assume that $a > b$. Write

$$1 + a = q(1+b) + r$$

with q and r integers satisfying $0 \leq r \leq b$. Then $m(a,b) = q$. After rearranging, the inequality

$$q \leq \frac{1+na}{W+nb}$$

is equivalent to

$$n \geq \frac{qW-1}{q+r-1}.$$

(Note that $q \geq 1$ and $r \geq 0$, and we cannot have equality in both of these inequalities as $a \neq b$, so the denominator is positive. This proves part (a)).

Note that when $q = 2$ and $r = 0$ the fraction on the right hand side is equal to $2W-1$, so $k(a,b) = 2W-1$ is attainable. It remains to show that $2W - 1 \geq \frac{qW-1}{q+r-1}$.

If $q \geq 2$ then

$$2W - 1 \geq W + \frac{W-1}{q-1} = \frac{qW-1}{q-1} \geq \frac{qW-1}{q+r-1}$$

as required.

If $q = 1$ then as $a \neq b$ we have $r \geq 1$ and hence

$$2W - 1 > W - 1 = \frac{qW-1}{q} \geq \frac{qW-1}{q+r-1},$$

as required.

This completes the proof that the maximum value asked for is $2W - 1$.

Solution 2

If $a < b$ then the sequence $\left\{\frac{1+na}{1+nb}\right\}_{n=1}^{\infty}$ is decreasing, and since the limit of the sequence is $\frac{a}{b}$, we have $m(a,b) = \lfloor \frac{a}{b} \rfloor = 0$ in this case.

On the other hand, if $a > b$, then the sequence $\left\{\frac{1+na}{1+nb}\right\}_{n=1}^{\infty}$ is increasing, so that $m(a,b) = \left\lfloor \frac{a+1}{b+1} \right\rfloor$ in this case.

(a) When $a < b$, we have that $m(a,b) = 0 \leq \frac{1+na}{W+nb}$ for all $n \geq 1$, hence $k(a,b) = 1$ in this case. Now assume that $a > b$. We want to show that $m(a,b) = \left\lfloor \frac{a+1}{b+1} \right\rfloor \leq \frac{1+na}{W+nb}$ for all $n \geq k$, for some $k \geq 1$. Solving the inequality gives

$$n \geq \frac{\left\lfloor \frac{a+1}{b+1} \right\rfloor W - 1}{a - \left\lfloor \frac{a+1}{b+1} \right\rfloor b}.$$

(Note that $a - \left\lfloor \frac{a+1}{b+1} \right\rfloor b > 0$ since $a > b$.) It follows that

$$k(a,b) = \left\lceil \frac{\left\lfloor \frac{a+1}{b+1} \right\rfloor W - 1}{a - \left\lfloor \frac{a+1}{b+1} \right\rfloor b} \right\rceil$$

when $a > b$.

(b) Since $k(a,b) = 1$ when $a < b$, we only have to consider the case $a > b$ if we want to find $\max\{k(a,b) \mid (a,b) \in S\}$. Using $\frac{a+1}{b+1} = \frac{a-b}{b+1} + 1$, we see that

$$k(a,b) = \left\lceil \frac{\left\lfloor \frac{a+1}{b+1} \right\rfloor W - 1}{a - \left\lfloor \frac{a+1}{b+1} \right\rfloor b} \right\rceil$$

$$= \left\lceil \frac{\left(\left\lfloor \frac{a-b}{b+1} \right\rfloor + 1\right) W - 1}{a - \left(\left\lfloor \frac{a-b}{b+1} \right\rfloor + 1\right) b} \right\rceil$$

$$= \left\lceil \frac{\left\lfloor \frac{a-b}{b+1} \right\rfloor W + W - 1}{a - b - \left\lfloor \frac{a-b}{b+1} \right\rfloor b} \right\rceil$$

Consider the following three possibilities:

I. $1 \le a - b < b + 1$. Here we have

$$k(a,b) = \left\lceil \frac{W-1}{a-b} \right\rceil \le W - 1.$$

II. $a - b = b + 1$. In this case,

$$k(a,b) = \left\lceil \frac{2W-1}{1} \right\rceil = 2W - 1.$$

III. $a - b > b + 1$. By the division algorithm, there exist $q \ge 1$ and $0 \le r < b + 1$ such that $a - b = q(b+1) + r$. Hence,

$$k(a,b) = \left\lceil \frac{qW + W - 1}{q + r} \right\rceil \le \left\lceil \frac{qW + W - 1}{q} \right\rceil = \left\lceil W + \frac{W-1}{q} \right\rceil \le 2W - 1.$$

Since I, II and III exhaust all cases, we conclude

$$\max\{k(a,b) \mid (a,b) \in S\} = 2W - 1.$$

This maximum value is obtained when $a = 2b + 1$.

Problem A4 (proposed by Ivan Guo and Ross Atkins)

Define a sequence by $s_0 = 1$ and for $d \geq 1$, $s_d = s_{d-1} + X_d$, where X_d is chosen uniformly at random from the set $\{1, 2, \ldots, d\}$.

What is the probability that the sequence $s_0, s_1, s_2, \ldots$ contains infinitely many primes?

Solution

Let $S = \{s_0, s_1, s_2, \ldots\}$. If T is a subset of the positive integers, we will say that T is *avoidable* if $P(S \cap T = \emptyset) > 0$.

Lemma. *Let T be a finite avoidable subset of the positive integers and let $n > 1$ be a positive integer that is larger than every element of T. Then*

$$P(n \in S \mid S \cap T = \emptyset) \geq \frac{1}{n}.$$

Proof. Since $1 \leq s_k - s_{k-1} \leq k$ for all k, we have

$$\begin{aligned} P(n \in S) &= \sum_{k=0}^{\infty} P(s_k = n) \\ &= \sum_{k=0}^{\infty} P(s_k = n \mid s_{k-1} \in [n-k, n-1])\, P(s_{k-1} \in [n-k, n-1]). \end{aligned}$$

The number s_k is chosen uniformly at random from $\{s_{k-1}+1, s_{k-1}+2, \ldots, s_{k-1}+k\}$. So for a fixed value of s_{k-1}, conditioning on the probability that $S \cap T \neq \emptyset$, then if $n \in [s_{k-1}+1, s_{k-1}+k]$, the probability that s_k is equal to n is given by

$$|\{s_{k-1}+1, s_{k-1}+2, \ldots, s_{k-1}+k\} \setminus T|^{-1} \geq k^{-1}.$$

Thus,

$$P(s_k = n \mid s_{k-1} \in [n-k, n-1], S \cap T = \emptyset) \geq \frac{1}{k}.$$

From the inequality $s_k \geq k$, we get $P(s_k = n) = 0$ for $k > n$ and hence

$$\begin{aligned} P(n \in S \mid S \cap T = \emptyset) &\geq \sum_{k=1}^{n} \frac{1}{k} P(s_{k-1} \in [n-k, n-1] \mid S \cap T = \emptyset) \\ &\geq \frac{1}{n} \sum_{k=1}^{n} P(s_{k-1} \in [n-k, n-1] \mid S \cap T = \emptyset) \end{aligned}$$

Let m be the smallest index such that $s_m \geq n$. Then as $s_m - s_{m-1} \leq m$, we get $n - 1 \geq s_{m-1} \geq n - m$. Hence the statement $s_{k-1} \in [n-k, n-1])$ must be true for at least one value of k. Thus the final sum of probabilities is at least one, proving $P(n \in S \mid S \cap T = \emptyset) \geq \frac{1}{n}$, as required. □

Let $p_1 < p_2 < p_3 < \cdots$ be the prime numbers. Fix an positive integer t. If $i > t$ is a positive integer, apply the lemma with $T = \{p_t, p_{t+1}, \ldots, p_{i-1}\}$ and $n = p_i$ to conclude that the probability that p_i is in S, conditional on the fact that $p_t, p_{t+1}, \ldots, p_{i-1}$ are all not in S, is at least p_i^{-1}. Hence the probability that $p_t, p_{t+1}, p_{t+2}, \ldots$ are all not in S is at most

$$\prod_{i=t}^{\infty} \left(1 - \frac{1}{p}\right).$$

This product is well-known to be zero, since the sum of the reciprocals of the primes diverges, using the inequality $1 - x \leq e^{-x}$ on each factor.

Now suppose that S contains finitely many primes. Then for some positive integer t, the numbers $p_t, p_{t+1}, p_{t+2}, \ldots$ do all not lie in S. Since probability is countably additive, this implies that the probability that S contains a finite number of primes is zero. Hence the probability that S contains infinitely many primes is equal to 1.

Problem B1 (proposed by Chris Tuffley)

Alice has six boxes labelled 1 through 6. She secretly chooses exactly two of the boxes and places a coin inside each. Bob is trying to guess which two boxes contain the coins. Each time Bob guesses, he does so by tapping exactly two of the boxes. Alice then responds by telling him the total number of coins inside the two boxes that he tapped. Bob successfully finds the two coins when Alice responds with the number 2.

What is the smallest positive integer n such that Bob can always find the two coins in at most n guesses?

Solution 1

We will show that the answer is 5. We will write $\{a, b\}$ to indicate the pair of boxes a and b.

First we give a strategy under which Bob can always find the two coins in at most five guesses. For the first four guesses, Bob guesses $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$.

If any of these guesses gets a response of 2, then Bob has already found the coins. Otherwise, at least one guess must have received a response of 1. If there was exactly one response of 1, say for the guess $\{1, n\}$, then $\{n, 6\}$ will be the fifth winning guess. If there were exactly two responses of 1, say for the guesses $\{1, m\}$ and $\{1, n\}$, then $\{m, n\}$ will be the fifth winning guess. If there were more than two responses of 1, then $\{1, 6\}$ will be the fifth winning guess.

We now show that a five-guess strategy is optimal. Since the only options on each turn are for Alice to receive an answer of 0, an answer of 1, or Bob finds the coins immediately, then forcing a win in at most n steps requires there to be at most $2^0+2^1+2^2+\cdots+2^{n-1} = 2^n-1$ possible game states. Initially there are $\binom{6}{2} = 15 = 2^4-1$ game states. So if we can win in four guesses, our first guess must either win, or narrow the number of game states down to 7 options. However, if the response to the first guess is 1, then there are $2 \times 4 = 8$ possible game states remaining. So we can never have a strategy that guarantees a win in at most four guesses.

Solution 2

We call the scenario in the problem the six-box game, and refer to the four-box game as the same scenario with the exception that the number of boxes is changed from six to four. We say that Bob wins when he finds the coins.

First we study the four-box game. We will show that Bob's optimal strategy in the four-box game's winning strategy requires up to 4 guesses.

Without loss of generality, say the first guess is $\{1, 2\}$. If the response is 2 then the game is won, and if the response is 0 then the second winning guess is $\{3, 4\}$. Otherwise, the response is 1, in which case guessing $\{3, 4\}$ will provide no additional information. So the second guess must include one box from $\{1, 2\}$ and one box from $\{3, 4\}$, which without loss of generality can be assumed to be $\{1, 3\}$. Again, if the response is 2 then the game is won, and if the response is 0 then the third winning guess is $\{2, 4\}$. Otherwise, the response is 1, in which case the remaining winning options are $\{1, 4\}$ or $\{2, 3\}$. These can be each guessed in turn, meaning at most 4 guesses are required.

Now consider the six-box game. Notice that if Alice's response to the first guess is 0, then we have reached the four-box game after 1 guess, meaning any strategy must require up to 5 guesses.

We now provide a strategy that guarantees winning in at most 5 guesses. First guess $\{1, 2\}$. If the response is 2, the game is won, and if the response is 0, play out the strategy for the four-box game on boxes 3, 4, 5 and 6. Otherwise, the response is 1, in which case we make the second guess $\{3, 4\}$. If the response is 2, the game is won in 2 guesses. If the response is 0, then we have made 2 guesses to reach a state equivalent to 1 guess into the four-box game on boxes 1, 2, 5 and 6. If the response is 1, then we have made 2 guesses to reach a state equivalent to 1 guess into the four-box game on boxes 1, 2, 3 and 4. So in either case, the game can be won in at most 3 more guesses on this four-box game.

Problem B2 (proposed by Johan Meyer)

Determine all continuous functions $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ that satisfy

$$f(x) = (x + 1)\, f(x^2),$$

for all $x \in \mathbb{R} \setminus \{-1, 1\}$.

Solution

Define $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ by $g(x) = (x - 1)f(x)$. Then for all $x \neq \pm 1$, we have

$$g(x) = (x - 1)f(x) = (x - 1)(x + 1)f(x^2) = (x^2 - 1)f(x^2) = g(x^2).$$

Note that this also implies $g(x) = g(-x)$ for $x \neq \pm 1$, i.e. g is even.

If $x \in [0, 1)$ then the sequence $\{x^{2^n}\}_{n=1}^{\infty}$ converges to 0. Since g is constant on this sequence, $g(x) = g(0)$ by the continuity of g. Since g is even and continuous, $g(x)$ is constant on $[-1, 1)$.

If $x > 1$ then the sequence $\{x^{1/2^n}\}_{n=1}^{\infty}$ converges to 1 and again g is constant on this sequence. Also $\{-x^{1/2^n}\}_{n=1}^{\infty}$ converges to -1 and g is equal to the same constant on this sequence since g is even. Therefore $g(x) = g(-1)$ by the continuity of g.

We have therefore shown g is constant, which implies that $f(x) = A/(x - 1)$ for some constant $A \in \mathbb{R}$. We can easily check that all such solutions satisfy the functional equation:

$$(x + 1)f(x^2) = (x + 1)\frac{A}{(x^2 - 1)} = \frac{A}{x - 1} = f(x).$$

Problem B3 (proposed by Andrew McGregor)

Let $\mathcal{L}$ be the set of all lines in the plane and let $\mathcal{P}$ be the set of all points in the plane.

Determine whether there exists a function $g : \mathcal{L} \to \mathcal{P}$ such that for any two distinct non-parallel lines $\ell_1, \ell_2 \in \mathcal{L}$, we have

(a) $g(\ell_1) \neq g(\ell_2)$, and

(b) if ℓ_3 is the line passing through $g(\ell_1)$ and $g(\ell_2)$, then $g(\ell_3)$ is the intersection of ℓ_1 and ℓ_2.

Solution 1

Suppose such a function g does exist.

First we show g is injective. Suppose a and b are distinct lines in $\mathcal{L}$ with $g(a) = g(b)$. Clearly by property (a), a and b must be parallel. Let c be any transversal line (not parallel to a and b). Then by property (a), $g(c)$ must be distinct from $g(a) = g(b)$. Let ℓ be the line through $g(a)$ and $g(c)$. Then by property (b), $g(\ell) = a \cap c$, and similarly since $g(a) = g(b)$, we must have $g(\ell) = b \cap c$. This implies $g(\ell)$ lies on both a and b, which is a contradiction since distinct parallel lines have no common point. Hence g is injective.

Now take any line ℓ and consider two distinct lines a and b through $g(\ell)$. By property (a), $g(a)$ and $g(b)$ are distinct, so let c be the line through $g(a)$ and $g(b)$. Then by property (b), $g(c)$ is the intersection of a and b, so $g(c) = g(\ell)$. By the injectivity of g, we have $c = \ell$. So for any line x through $g(\ell)$, we know $g(x)$ lies on ℓ.

Now let a and b be two distinct parallel lines. Since $g(a)$ and $g(b)$ are distinct by the injectivity of g, let ℓ be the line through $g(a)$ and $g(b)$. Then by the above result, $g(\ell)$ lies on both a and b, which is a contradiction since parallel lines have no common point.

So there can be no such function g.

Solution 2

Suppose such a function g does exist.

Let a and b be any two non-parallel lines in $\mathcal{L}$. Then by property (a), $g(a)$ and $g(b)$ are distinct, so let c be the line through $g(a)$ and $g(b)$. By property (b), $g(c) = a \cap b$. Since at least one of a and b is not parallel to c, suppose without loss of generality that $a \nparallel c$.

We first claim that a and b can be chosen so that c is not concurrent with a and b. If this were not possible, then necessarily every pair of non-parallel lines ℓ_1 and ℓ_2 would have the line through $g(\ell_1)$ and $g(\ell_2)$ go through $\ell_1 \cap \ell_2$. Then consider any line x that crosses c once at some point other than $g(c)$. We require that $g(x)$ lies on c for the rule to hold, but then since c goes through $g(c)$ and $g(x)$, by property (b) we must have that $g(c) = c \cap x$, which contradicts that x does not intersect c at $g(c)$.

We now claim that any point P on c is the image under g of some line $y \nparallel c$ through $g(c)$. Given any point P on c, let x be the line through P and $g(c)$. Note that by property (a),

$g(x) \neq g(c)$. So letting y be the line through $g(x)$ and $g(c)$, we have $P = x \cap c = g(y)$ by property (b). We now show that $y \nparallel c$ by showing $g(x)$ lies on c. First, if $g(x) = g(a)$ then clearly $g(x)$ lies on c. Otherwise, g maps the line through $g(a)$ and $g(x)$ to $a \cap x = g(c)$ by property (b). But the line c is the only line through $g(a)$ that maps to $g(c)$ by property (a). So $g(x)$ lies on c.

Now let ℓ be the line through $g(c)$ that is parallel to c. Every point on c is already the image of some line $y \nparallel c$ through $g(c)$, so by property (a), $g(\ell)$ cannot lie on c. In particular, $g(\ell) \neq g(a)$, so the line through $g(a)$ and $g(\ell)$ maps to $a \cap \ell = g(c)$ by property (b). But again, the line c is the only line through $g(a)$ that maps to $g(c)$ by property (a), implying $g(\ell)$ lies on c, which is a contradiction.

So there can be no such function g.

Problem B4 (proposed by Peter McNamara)

Let n be an odd positive integer and let

$$f_n(x, y, z) = x^n + y^n + z^n + (x + y + z)^n.$$

(a) Prove that there exist infinitely many values of n such that

$$f_n(x, y, z) \equiv (x + y)(y + z)(z + x)\, g(x, y, z)\, h(x, y, z) \pmod 2,$$

for some integer polynomials $g(x, y, z)$ and $h(x, y, z)$, neither of which is constant modulo 2.

(b) Determine all values of n such that

$$f_n(x, y, z) \equiv (x + y)(y + z)(z + x)\, g(x, y, z)\, h(x, y, z) \pmod 2,$$

for some integer polynomials $g(x, y, z)$ and $h(x, y, z)$, neither of which is constant modulo 2.

(Two integer polynomials are *congruent modulo 2* if every coefficient of their difference is even. A polynomial is *constant modulo 2* if it is congruent to a constant polynomial modulo 2.)

Solution 1 to part (a)

Let $m \geq 3$ be an integer and let $q = 2^m$. Let $n = q + 1$. We will provide a factorisation whenever $q - 1$ is composite, which occurs for infinitely many values of n, for example whenever m is even, $q - 1$ is divisible by 3. We work modulo 2 throughout, which implies that we can use the identities $(a + b)^2 = a^2 + b^2$ and $(a + b)^q = a^q + b^q$, the latter of which is obtained by applying the former m times. We will often use these identities without comment.

Since f_n is homogeneous, we may assume without loss of generality that $z = 1$. Now make the substitution $x = s + 1, y = t + 1$. With this substitution our polynomial simplifies to

$$f_n(x, y, z) = (s + 1)(s^q + 1) + (t + 1)(t^q + 1) + 1 + (s + t + 1)(s^q + t^q + 1) = st(s^{q-1} - t^{q-1}).$$

Assuming $q - 1$ is composite, write $q - 1 = ab$ with a and b integers larger than 1. Then

$$s^{q-1} - t^{q-1} = (s^a)^b - (t^a)^b = (s^a - t^a)(s^{ab-b} + s^{ab-2b}t^b + \cdots + t^{ab-b}).$$

Hence both $(s^a - t^a)/(s - t)$ and $s^{ab-b} + s^{ab-2b}t^b + \cdots + t^{ab-b}$ are factors, rehomogenising we get a factorisation with

$$g(x, y, z) = \frac{(x + z)^a - (y + z)^a}{x - y}$$

and

$$h(x, y, z) = (x + z)^{ab-b} + (x + z)^{ab-2b}(y + z)^b + \cdots + (y + z)^{ab-b}.$$

As $q-1$ is odd, a is at least 3 and therefore these polynomials are nonconstant.

Solution 2 to part (a)

Let $m \geq 3$ be an integer and let $n = 2^m + 1$. Let $q = n - 1 = 2^m$. We work throughout over the field $\mathbb{F}_q$ with q elements, and frequently use the identity $(a+b)^2 = a^2 + b^2$ without comment. In particular applying this m times gives the identity $(a+b)^q = a^q + b^q$. We compute

$$\begin{aligned} f_n(x,y,z) &= x^{q+1} + y^{q+1} + z^{q+1} + (x+y+z)(x^q + y^q + z^q) \\ &= xy^q + yx^q + yz^q + zy^q + zx^q + xz^q \\ &= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^q & y^q & z^q \end{vmatrix} \end{aligned}$$

Suppose that $\alpha, \beta, \gamma \in \mathbb{F}_q$ are not all zero and satisfy $\alpha + \beta + \gamma = 0$.

If $\alpha x + \beta y + \gamma z = 0$ then $\alpha^q x^q + \beta^q y^q + \gamma^q z^q = 0$. Since $\alpha, \beta, \gamma \in \mathbb{F}_q$ we have $\alpha^q = \alpha$, $\beta^q = \beta$ and $\gamma^q = \gamma$, so $\alpha x^q + \beta y^q + \gamma z^q = 0$. Therefore

$$\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^q & y^q & z^q \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0$$

and hence $f_n(x,y,z) = 0$ because of its incarnation as a determinant.

This proves that for all $\alpha, \beta, \gamma \in \mathbb{F}_q$ that are not all zero and satisfy $\alpha + \beta + \gamma = 0$, the polynomial $\alpha x + \beta y + \gamma z$ divides $f_n(x,y,z)$ (note it is important here that we are not merely specialising x, y and z to elements of $\mathbb{F}_q$).

For such a choice of α, β and γ as above, define

$$g(x,y,z) = \prod_{i=0}^{m-1} (\alpha^{2^i} x + \beta^{2^i} y + \gamma^{2^i} z).$$

We first show that $g(x,y,z) \in \mathbb{F}_2[x,y,z]$. To achieve this we will introduce the operator $F : \mathbb{F}_q[x,y,z] \to \mathbb{F}_q[x,y,z]$ defined by

$$F\left(\sum_{i,j,k} a_{ijk} x^i y^j z^k\right) = \sum_{i,j,k} a_{ijk}^2 x^i y^j z^k.$$

Note that for any polynomials p_1 and p_1 we have $F(p_1 p_2) = F(p_1)F(p_2)$ and $F(p_1 + p_2) = F(p_1) + F(p_2)$.

Since $\alpha^q = \alpha$, $\beta^q = \beta$ and $\gamma^q = \gamma$, we get that $F(g) = g$. Therefore every coefficient of g is equal to its square, hence $g(x,y,z) \in \mathbb{F}_2[x,y,z]$, as required.

Since $\mathbb{F}_q[x,y,z]$ is a unique factorisation domain and each irreducible factor of $g(x,y,z)$ divides $f_n(x,y,z)$, the polynomial $g(x,y,z)$ will divide $f_n(x,y,z)$ if none of its irreducible factors are scalar multiples of one another. We can achieve this by choosing $\alpha \in \mathbb{F}_q$ to be a primitive root, setting $\beta = \alpha + 1$ and $\gamma = 1$. In particular this choice implies that the powers α^{2^i} for $0 \le i \le m-1$ are all distinct, hence the factors of $g(x,y,z)$ are relatively prime, as required.

As $m \ge 3$, $\deg(f_n) > 3 + \deg(g)$ and so the complementary factor

$$h(x,y,z) = \frac{f_n(x,y,z)}{(x+y)(y+z)(z+x)g(x,y,z)}$$

is also nonconstant, so we have shown the existence of the desired factorisation.

Known partial results for part (b)

Write

$$p_n(x,y,z) = \frac{f_n(x,y,z)}{(x+y)(y+z)(z+x)}$$

If $n = 4^m - 2^m + 1$ with $m \ge 4$, then it is known that $p_n(x,y,z)$ is reducible.

When $n \equiv 3 \pmod 4$ then it is known that $p_n(x,y,z)$ is irreducible.

These results are due to Janwa and Wilson [1], and to Janwa, McGuire and Wilson [2] respectively, who came across this problem in studying error correcting codes. It was further conjectured in [2] that the only odd values of n for which $p_n(x,y,z)$ is reducible are the two families $n = 2^m + 1$ and $n = 4^m - 2^m + 1$, but this was shown to be false by Hernando and McGuire [3], who found a factorisation of $p_{205}(x,y,z)$.

[1] H. Janwa and R. M. Wilson, *Hyperplane sections of Fermat varieties in P^3 in char. 2 and some applications to cyclic codes*, (Applied Algebra, Algebraic Algorithms and Error-Correcting Codes. Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, San Juan, PR, 1993. Applied Algebra, Algebraic Algorithms and Error-Correcting Codes. Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, San Juan, PR, 1993, Lecture Notes in Comput. Sci., vol. 673 (1993), Springer: Springer Berlin), 180-194

[2] H. Janwa, G. McGuire and R. M. Wilson, *Double-error-correcting cyclic codes and absolutely irreducible polynomials over GF(2)*, J. Algebra, 178, 2, 665-676 (1995)

[3] F. Hernando and G. McGuire, *Proof of a conjecture on the sequence of exceptional numbers, classifying cyclic codes and APN functions*, J. Algebra 343, 78-92 (2011).

Problem C1 (proposed by Ian Payne)

We say that a square in the plane is *centred* if the intersection of its diagonals is at the origin.

Prove that there exists a centred square with exactly d lattice points in its interior if and only if d is a positive integer satisfying $d \equiv 1 \pmod 4$.

(A *lattice point* is a point in $\mathbb{R}^2$ whose coordinates are both integers).

Solution

Pick an irrational number α. Let S_x be the centred square with side lengths x and one side having slope α. Let $f(x)$ be the number of lattice points in the interior of S_x.

For a sufficiently small positive real number x, $f(x) = 1$.

The function $f(x)$ is a non-decreasing function from $(0, \infty)$ to $\mathbb{Z}$. Let us consider what happens at a point of discontinuity $x = x_0$. This means that S_{x_0} has at least one lattice point on its boundary. The boundary of S_{x_0} is invariant under rotation by 90 degrees, as is the set of lattice points. Therefore the number of lattice points on the boundary of S_{x_0} is at least 4. Suppose that there are at least 5 lattice points on the boundary of S_{x_0}. Then there would be two lattice points on the same side. This forces the gradient of that side to be rational, contradicting the fact that α is irrational.

Hence $f(x)$ jumps by four at every point of discontinuity, which shows that for every integer $d \equiv 1 \pmod 4$, there exists a centred square with exactly d lattice points in its interior.

For the converse direction, suppose that a centred square has exactly d lattice points in its interior. Consider the action of rotation by 90 degrees. This rotation preserves the set of lattice points in a centred square. There is one fixed point under this action, namely the origin, and every other lattice point lies in an orbit of four elements. Hence $d \equiv 1 \pmod 4$.

Problem C2 (proposed by Ian Payne)

Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^3 - 3x^2 + \frac{7}{2}x - \frac{1}{2}.$$

Determine all real numbers r such that $f(f(r)) = r$.

Solution 1

We compute

$$f'(x) = 3x^2 - 6x + \frac{7}{2} > 3(x-1)^2 \geq 0.$$

Hence $f(x)$ is a strictly increasing function.

Now if $f(f(r)) = r$ and $r < f(r)$, then since f is increasing we get $f(r) < f(f(r)) = r$, a contradiction. A similar contradiction is reached if $f(r) > r$.

Hence the only solutions to $f(f(r)) = r$ are solutions to $f(r) = r$. There are at most three real solutions since $f(x)$ is a cubic. We can check that 1, $\frac{2-\sqrt{2}}{2}$ and $\frac{2+\sqrt{2}}{2}$ are all solutions, so this constitutes the entire solution set.

Solution 2

Consider the polynomial $g(x) = f(x) - x = x^3 - 3x^2 + \frac{5}{2}x - \frac{1}{2} = (x-1)(x^2 - 2x + \frac{1}{2})$. The three roots of $g(x)$ are 1 and $\frac{2\pm\sqrt{2}}{2}$. Let $a = \frac{2-\sqrt{2}}{2}$, $b = 1$, and $c = \frac{2+\sqrt{2}}{2}$ so that a, b, and c are the roots of $g(x)$, and observe that $a, b, c \in (0, 2)$.

A root r of $g(x)$ has the property that $f(r) - r = 0$, so $f(r) = r$. Therefore, the three roots of g have the property that $f(f(r)) = f(r) = r$. We will show that no real numbers other than the three roots of $g(x)$ have the desired property.

Since $g(x) = (x-a)(x-b)(x-c)$, we have that $f(f(x)) - x$ is

$$\begin{aligned}
&\big(f(x) - a\big)\big(f(x) - b\big)\big(f(x) - c\big) + f(x) - x \\
&= \big(g(x) + x - a\big)\big(g(x) + x - b\big)\big(g(x) + x - c\big) + g(x) + x - x \\
&= (x-a)\big((x-b)(x-c) + 1\big)(x-b)\big((x-a)(x-c) + 1\big)(x-c)\big((x-a)(x-b) + 1\big) + g(x) \\
&= g(x)\Big(\big((x-b)(x-c) + 1\big)\big((x-a)(x-c) + 1\big)\big((x-a)(x-b) + 1\big) + 1\Big),
\end{aligned}$$

It remains to show that the polynomial

$$\big((x-b)(x-c) + 1\big)\big((x-a)(x-c) + 1\big)\big((x-a)(x-b) + 1\big) + 1$$

has no real roots.

For two fixed real numbers u and v, the expression $(x-u)(x-v)$ is minimised when $x = \dfrac{u+v}{2}$, so the minimum of $(x-u)(x-v) + 1$ is

$$\left(\frac{u+v}{2} - u\right)\left(\frac{u+v}{2} - v\right) + 1 = \frac{4 - (v-u)^2}{4}$$

and this expression will be positive provided $(v-u)^2 < 4$ or $|v-u| < 2$. Since $a, b, c \in (0, 2)$, we have $|c-b|, |a-c|, |b-a| < 2$. Therefore

$$\big((x-b)(x-c)+1\big)\big((x-a)(x-c)+1\big)\big((x-a)(x-b)+1\big)+1$$

is strictly positive and hence has no real roots, as required.

Solution 3

As in the other solutions, 1 and $\frac{2\pm\sqrt{2}}{2}$ have the given property. Suppose now that $f(f(p)) = p$ but $f(p) \neq p$. Let $q = f(p)$. Then $f(f(p)) = p$ implies $f(q) = p$. This leads to the equations

$$\begin{aligned} p^3 - 3p^2 + \frac{7}{2}p - \frac{1}{2} &= q \\ q^3 - 3q^2 + \frac{7}{2}q - \frac{1}{2} &= p \end{aligned}$$

which can be subtracted to get

$$(p^3 - q^3) - 3(p^2 - q^2) + \frac{7}{2}(p-q) = q - p.$$

Since $p \neq q$, we can divide by $p - q$ to get

$$p^2 + pq + q^2 - 3(p+q) + \frac{9}{2} = 0.$$

Considered as a quadratic in p, the discriminant of this equation is

$$(q-3)^2 - 4\left(q^2 - 3q + \frac{9}{2}\right) = -3q^2 + 6q - 9 = -3\big((q-1)^2 + 2\big)$$

which is always negative. Therefore, there are no real solutions, so there are no real numbers r such that $f(f(r)) = r$ other than those for which $f(r) = r$.

Problem C3 (proposed by Quang Ong)

For each positive integer n, let a_n be the number of integers k such that $0 \le k \le n$ and the binomial coefficient $\binom{n}{k}$ is odd.

For which $s \in [1, \infty)$ does

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converge?

Solution 1

If we consider the entries in Pascal's triangle (an array of binomial coefficients $\binom{n}{k}$ with rows indexed by n) reduced modulo 2, a pattern emerges reminiscent of the Sierpinski triangle fractal. Specifically, for each non-negative integer r, row $2^r - 1$ of Pascal's triangle contains only 1s, and the array from rows 2^r to $2^{r+1} - 1$ appears as two copies of the array from rows 0 to $2^r - 1$ side by side, with 0 entries filling the space between them. This can easily be shown by induction: it clearly holds for the case $r = 0$, and assuming it holds when $r = a$, we note that in particular the two copies of row $2^a - 1$ side by side in row $2^{a+1} - 1$ must be a row of only 1s. Furthermore, by Pascal's rule $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, we know that all entries in row 2^{a+1} except for the first and last must be 0 since they are each the sum of a pair of 1s. For the rows that proceed from here, the single 1 on each end of row 2^{a+1} generates the same pattern as the top 1 did in row 0, with no interference between the patterns until they meet at row $2^{a+2} - 1$.

So the number of 1s from row 0 to row $2^r - 1$ is 3 times the number of 1s from row 0 to row $2^{r-1} - 1$, meaning the number of 1s from row 0 to row $2^r - 1$ is 3^r. Thus the number of 1s from row 2^{r-1} to row $2^r - 1$ is $\frac{2}{3} \cdot 3^r$. Let

$$A_r = \sum_{n=2^{r-1}}^{2^r-1} \frac{a_n}{n^s}.$$

Then

$$A_r = \sum_{n=2^{r-1}}^{2^r-1} \frac{a_n}{n^s} \ge \sum_{n=2^{r-1}}^{2^r-1} \frac{a_n}{(2^r)^s} = \frac{2}{3}\left(\frac{3}{2^s}\right)^r,$$

so $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{r=1}^{\infty} A_r$ diverges whenever $\frac{3}{2^s} \ge 1$, that is, whenever $s \le \log_2(3)$.

We also have the upper bound

$$A_r = \sum_{n=2^{r-1}}^{2^r-1} \frac{a_n}{n^s} \le \sum_{n=2^{r-1}}^{2^r-1} \frac{a_n}{(2^{r-1})^s} = 2\left(\frac{3}{2^s}\right)^{r-1},$$

so $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{r=1}^{\infty} A_r$ converges whenever $\frac{3}{2^s} < 1$, that is, whenever $s > \log_2(3)$.

Combining these results, we see that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges precisely when $s > \log_2(3)$.

Solution 2

We will use the following well-known fact:

The binomial coefficient $\binom{a+b}{a}$ is odd if and only if there are no carries when a and b are added in base 2.

Define

$$X_m = \{(a,b) \in \mathbb{N}^2 \mid 2^{m-1} \le a+b < 2^m, \binom{a+b}{a} \equiv 1 \pmod 2\}.$$

We will show that $|X_m| = 2 \times 3^{m-1}$. Consider the numbers a and b written in binary. Since no carries can occur, the choices for any individual binary digit of a and b are (a) 0 and 0, (b) 0 and 1, and (c) 1 and 0. The most significant digit is the m-th digit, since $2^{m-1} \le a+b < 2^m$. In this case, case (a) is not possible or $a+b$ would be too small (remembering that no carries can occur).

Hence there are 2 possibilities for the m-th digits of a and b, and 3 possibilities for each of the remaining $m-1$ digits. These choices are independent so we get our desired formula.

We can write the sum as

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{m=1}^{\infty} \sum_{(a,b)\in X_m} \frac{1}{(a+b)^s}$$

From the bounds $2^{m-1} \le a+b < 2^m$ and the formula $|X_m| = 2 \times 3^{m-1}$, we get

$$\sum_{m=1}^{\infty} \frac{2 \times 3^{m-1}}{(2^m)^s} < \sum_{n=1}^{\infty} \frac{a_n}{n^s} < \sum_{m=1}^{\infty} \frac{2 \times 3^{m-1}}{(2^{m-1})^s}.$$

Both the lower and upper bounds are geometric series with ratio $\frac{3}{2^s}$ so converge if and only if this ratio is less than 1.

Therefore the sum converges if and only if $s > \log_2(3)$.

Problem C4 (proposed by Tony Guttman)

A queue of students is waiting to enter a lecture theatre that contains N seats in a row. Each student enters the lecture theatre and sits in a seat chosen uniformly at random from those seats not next to an already occupied seat. The lecturer declares the lecture theatre to be full when no more students can be seated in this manner. Let S_N denote the expected proportion of seats that are occupied when the lecture theatre is declared full.

Determine $\lim_{N\to\infty} S_N$ or prove that the limit does not exist.

Solution

Let E_N be the expected value of the number of students when the theatre is full. It will be convenient for us to say that $E_N = 0$ for $N \leq 0$. Our goal is to compute $\lim_{N\to\infty} \frac{E_N}{N}$.

Suppose $n \geq 0$ and consider the case $N = n+1$. If the first student sits in the $(i+1)$-st seat, then the seating rules force the i-th and $(i+2)$-nd seats to never be occupied. Removing these three seats breaks the lecture theatre into two smaller ones, with $i-1$ and $n-i-1$ seats respectively, which do not interact with each other. Thus we obtain the recurrence

$$E_{n+1} = 1 + \frac{1}{n+1}\sum_{i=0}^{n}(E_{i-1} + E_{n-1-i}) = 1 + \frac{2}{n+1}\sum_{i=0}^{n-1} E_i.$$

Rewrite this as

$$(n+1)E_{n+1} = n+1+2\sum_{i=0}^{n-1} E_i$$

and subtract it from

$$(n+2)E_{n+2} = n+2+2\sum_{i=0}^{n} E_i$$

to obtain the recurrence

$$(n+2)E_{n+2} - (n+1)E_{n+1} = 1 + 2E_n.$$

Define the generating function

$$E(x) = \sum_{n=0}^{\infty} E_n x^n.$$

Multiply the recurrence by x^n and sum over all $n \geq 0$ to obtain

$$\frac{E'(x)-1}{x} - E'(x) = 2E(x) + \frac{1}{1-x}.$$

This rearranges to

$$(1-x)E'(x) - 2xE(x) = \frac{1}{1-x}.$$

The integrating factor for this differential equation is $(x-1)^2e^{2x}$, so we can multiply through by $(1-x)e^{2x}$ to obtain

$$\frac{d}{dx}\left((x-1)^2e^{2x}E(x)\right) = e^{2x}$$

and since $E(0)=0$ this has solution

$$(x-1)^2e^{2x}E(x) = e^{2x}-1.$$

So

$$E(x) = \frac{1-e^{-2x}}{2(1-x)^2}.$$

Hence we have

$$\begin{aligned} 2E(x) &= \frac{1}{(1-x)^2} - \frac{e^{-2x}}{(1-x)^2} \\ &= \sum_{l=0}^{\infty}(l+1)x^l - \left(\sum_{k=0}^{\infty}\frac{(-2x)^k}{k!}\right)\left(\sum_{l=0}^{\infty}(l+1)x^l\right) \end{aligned}$$

Taking the coefficient of x^n, we have

$$2E_n = n+1-\sum_{n=0}^{n}\frac{(-2)^k}{k!}(n+1-k).$$

Dividing by n and rearranging gives

$$\frac{2E_n}{n} = \frac{n+1}{n}\left(1-\sum_{k=0}^{n}\frac{(-2)^k}{k!}\right)+\frac{1}{n}\left(\sum_{k=1}^{n}\frac{(-2)^k}{(k-1)!}\right).$$

The sum $\displaystyle\sum_{k=1}^{\infty}\frac{(-2)^k}{(k-1)!}$ is absolutely convergent, hence

$$\lim_{n\to\infty}\frac{2E_n}{n} = \lim_{n\to\infty}\left(1-\sum_{k=0}^{n}\frac{(-2)^k}{k!}\right).$$

This limit is $1-e^{-2}$, and therefore the answer to the problem is

$$\frac{1-e^{-2}}{2}.$$

Remark: Instead of giving an explicit formula for E_n, one can also invoke some complex analysis and use the fact that $E(x)$ is meromorphic with only a pole at $x=1$. From this observation, a study of the leading term in the Laurent expansion at $x=1$ gives the desired limiting behaviour.